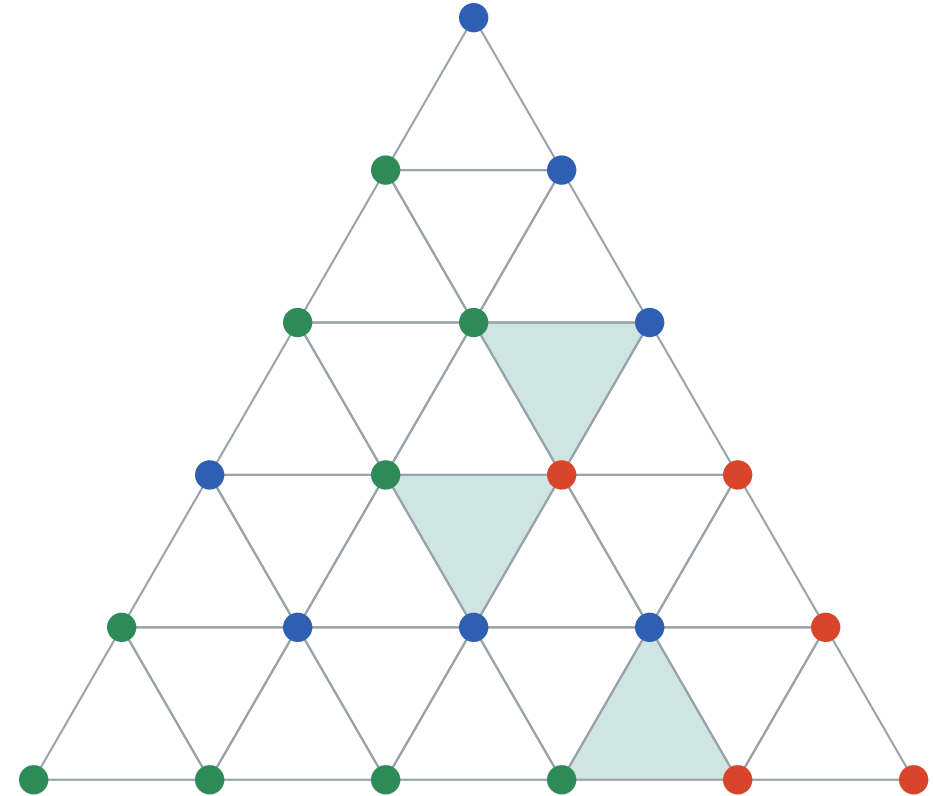


Fair Division

Lecture 02:
Envy-free Cake Cutting
and Sperner's Lemma

Hannaneh Akrami



Recap

Proportional

$$\forall i : v_i(A_i) \geq \frac{1}{n}$$

Steinhaus 1948

Envy-free (EF)

$$\forall i, j : v_i(A_i) \geq v_i(A_j)$$

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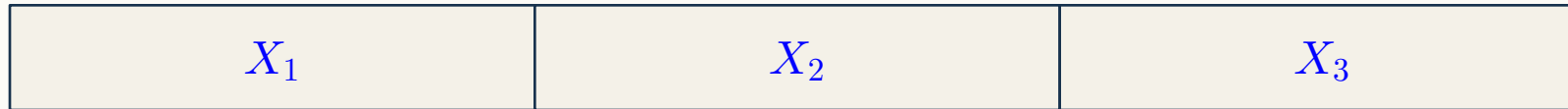
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EF for $n \geq 3$?

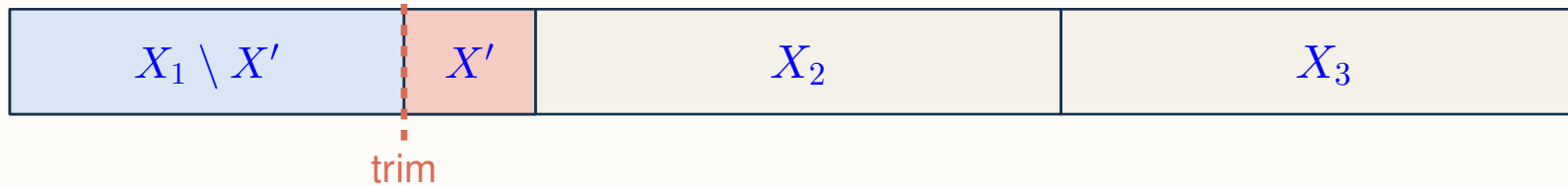
Selfridge–Conway (1960): EF for 3 Agents



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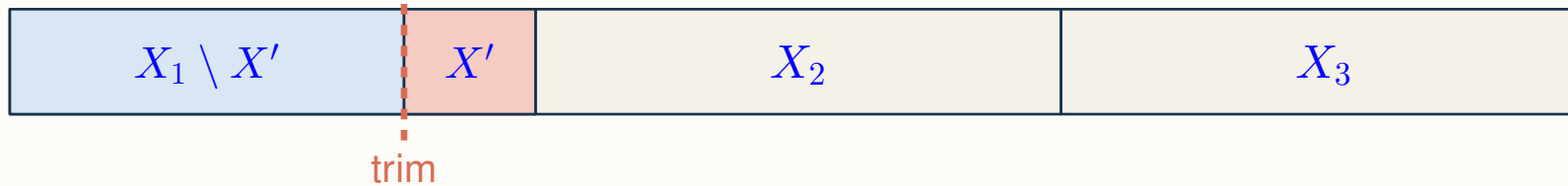
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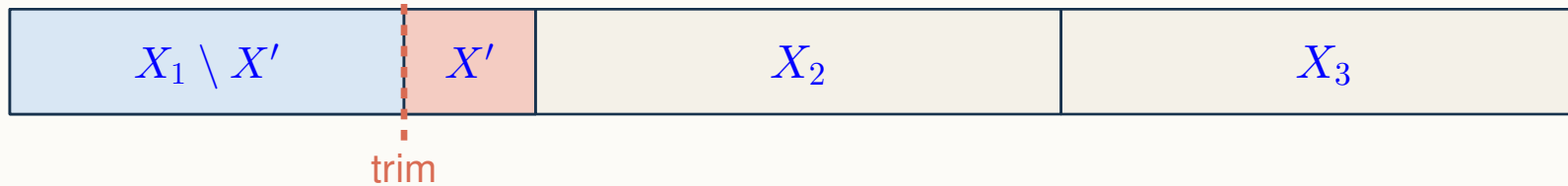
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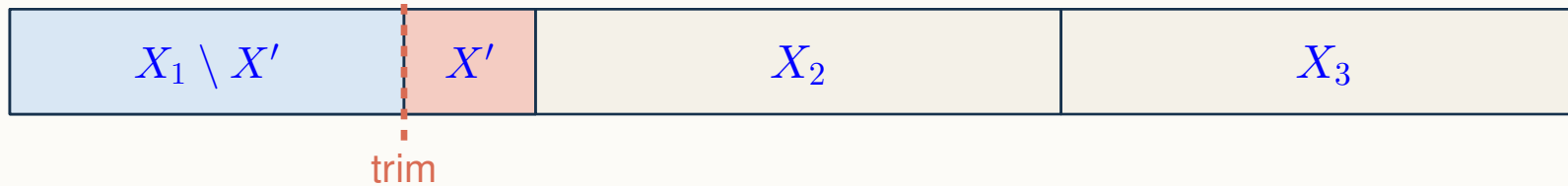
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Selfridge–Conway: Why Nobody Envies

Stage 1: 1 cuts in three, 2 trims X' , pick $3 \rightarrow 2 \rightarrow 1$. Stage 2: $\bar{T}$ cuts X' in three, pick $T \rightarrow 1 \rightarrow \bar{T}$.

Each share = stage-1 piece $\cup$ stage-2 piece; valuations are additive $\Rightarrow$ check each stage.

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Overall, T holds $(X_1 \setminus X') \cup Y$ with $Y \subseteq X'$:

$$v_1((X_1 \setminus X') \cup Y) \leq v_1(X_1) = \frac{1}{3} \leq v_1(\text{agent 1's share})$$

Selfridge–Conway: Properties

Properties:

Proportional

Envy-free

Equitable

Contiguous

Pareto optimal

Non-wasteful

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14

Robertson–Webb queries

≤ 5

cuts

2 + 3 + 3 + 2 + 4 queries; 2 cuts by agent 1, 1 trim, 2 on the trimmings

Envy-free for n Agents

Brams–Taylor (1995), Robertson–Webb (1997): **finite but unbounded** protocols

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Question. How far apart are the best upper and lower bounds for EF?

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Huge gap: closing it is a major open problem

Contiguous Pieces

Question. Selfridge–Conway gives non-contiguous pieces. Can a finite Robertson–Webb protocol always find an EF allocation with **contiguous** pieces?

Contiguous Pieces

Theorem (Stromquist 2008). For $n \geq 3$, no finite Robertson–Webb protocol guarantees an envy-free allocation with contiguous pieces.

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Question. Do envy-free contiguous allocations at least **exist**?

Contiguous Pieces

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Theorem (Stromquist 1980; Su 1999). For every n , there **exists** an envy-free and contiguous allocation of the cake.

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Theorem (Stromquist 1980; Su 1999). For every n , there **exists** an envy-free and contiguous allocation of the cake.

Proof idea (Simmons, popularised by Su): a combinatorial lemma that, on the face of it, has nothing to do with cake cutting ...

Sperner's lemma

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Sperner's Lemma



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Question. Why is this true?

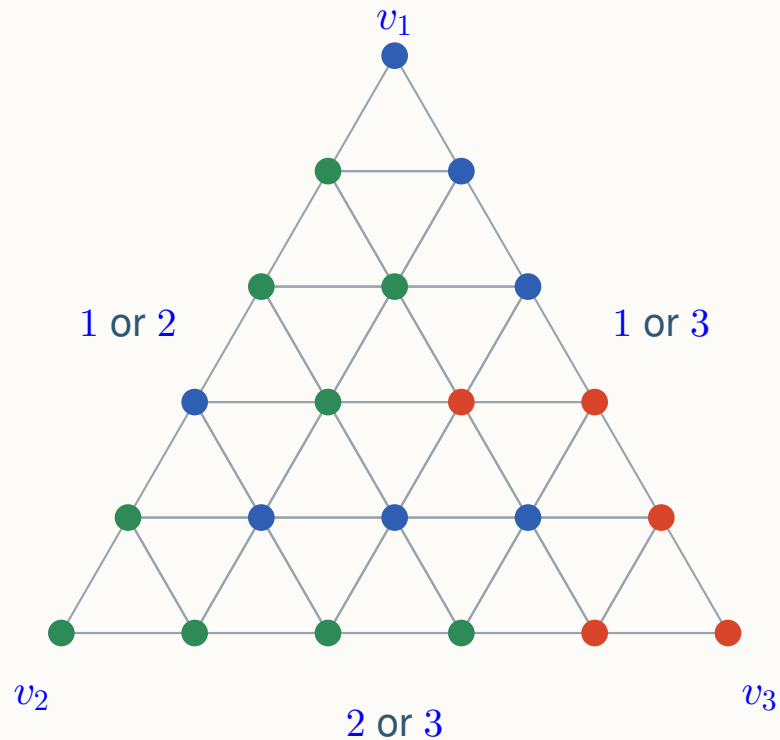
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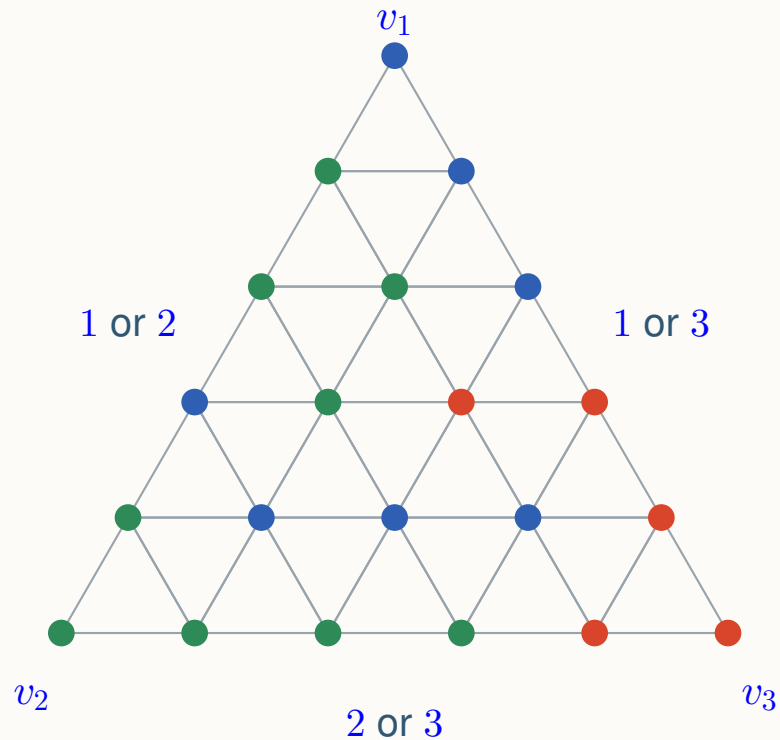


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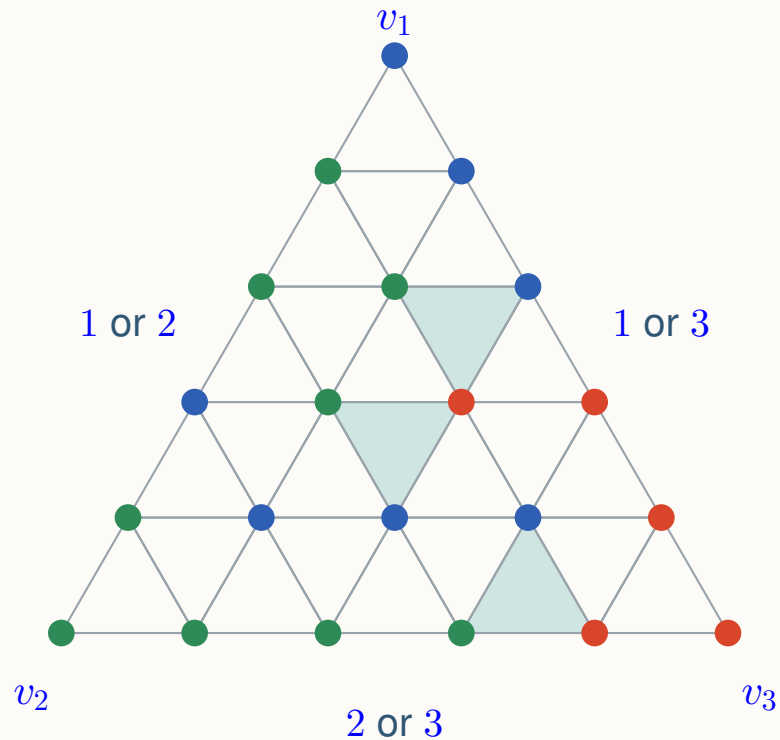
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Question. Must a Sperner-coloured triangulation contain a **fully coloured** baby triangle?

Sperner's Lemma



3 fully coloured (shaded)



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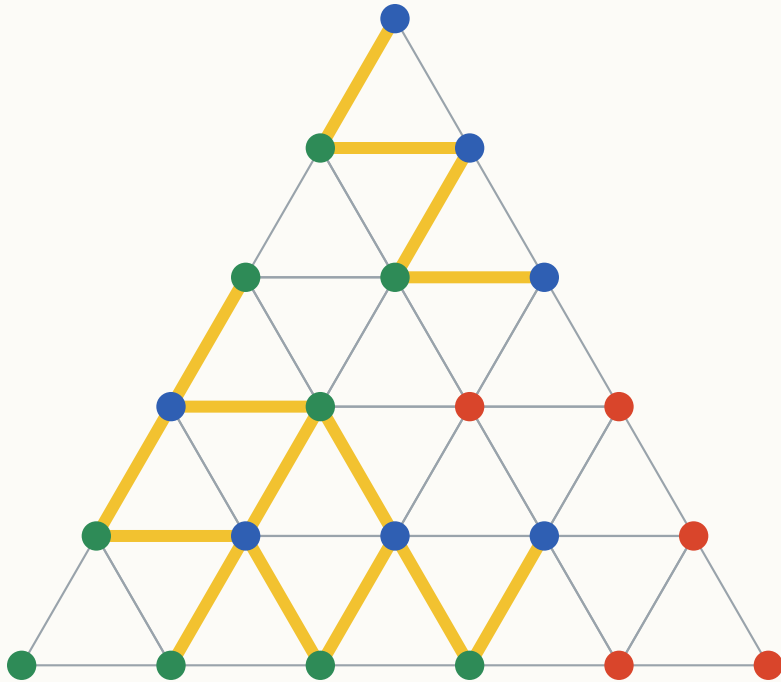
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Theorem (Sperner's lemma, 2D). Every Sperner-coloured triangulation has a fully coloured baby triangle; in fact an **odd** number of them.

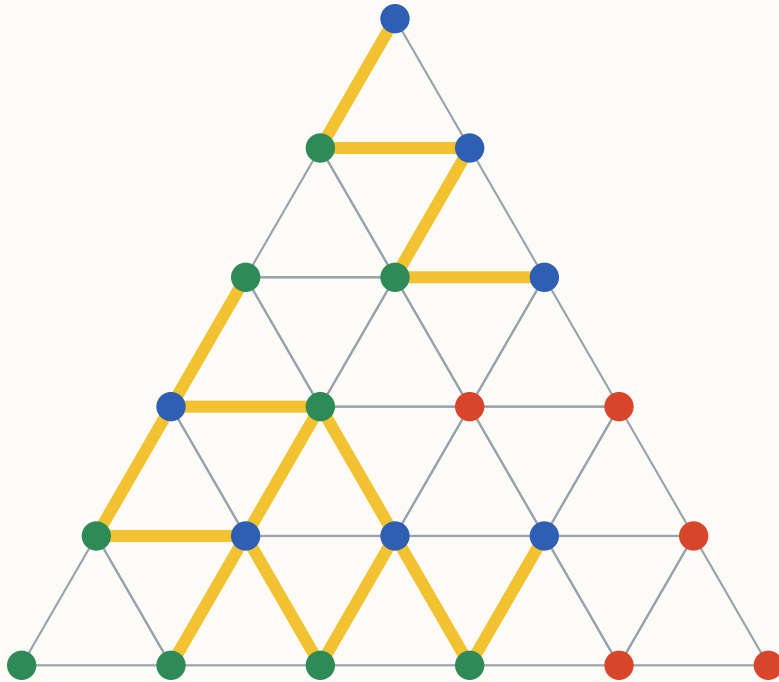
Proof: Walking Through Doors

House = big triangle, rooms = baby triangles, **door** = edge coloured $\{1, 2\}$



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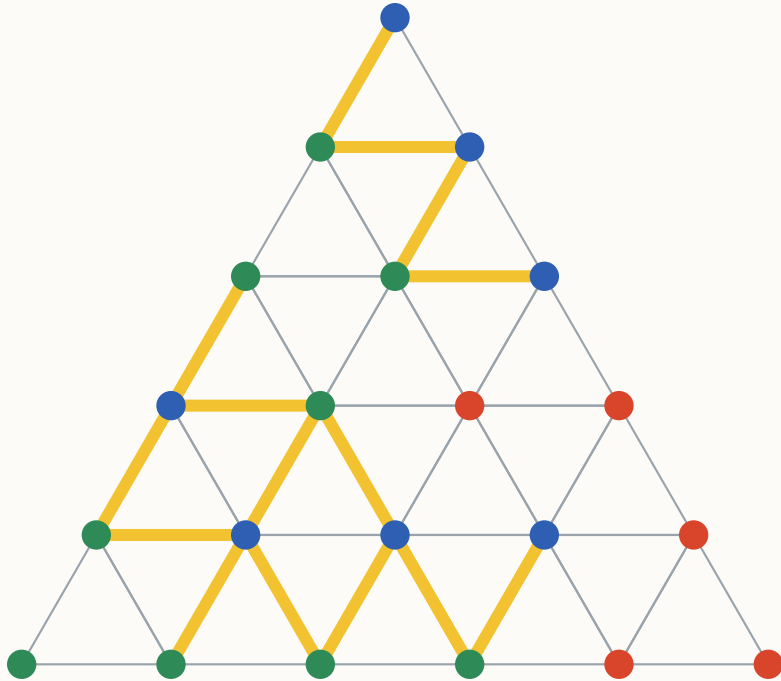
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Question. Where on the boundary can doors lie, and how many are there?

Proof: Walking Through Doors

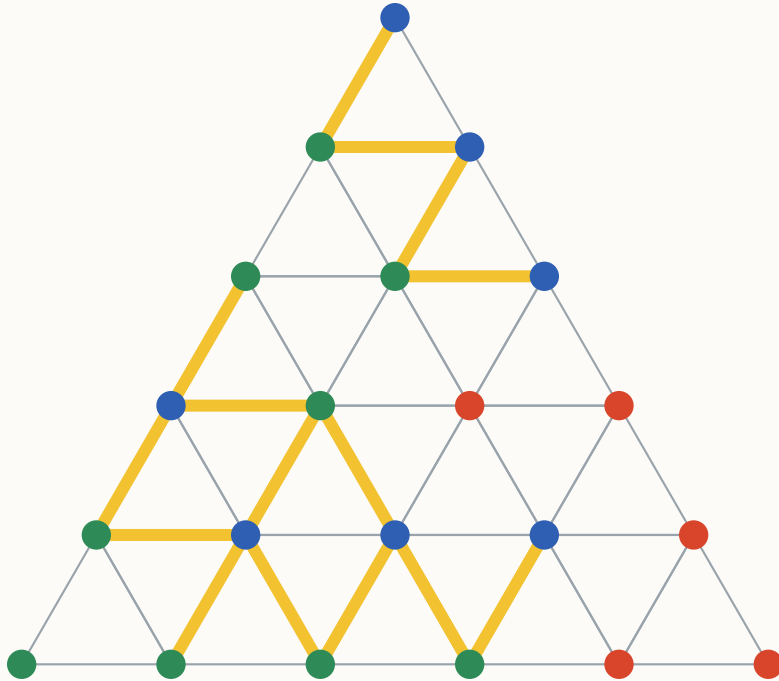
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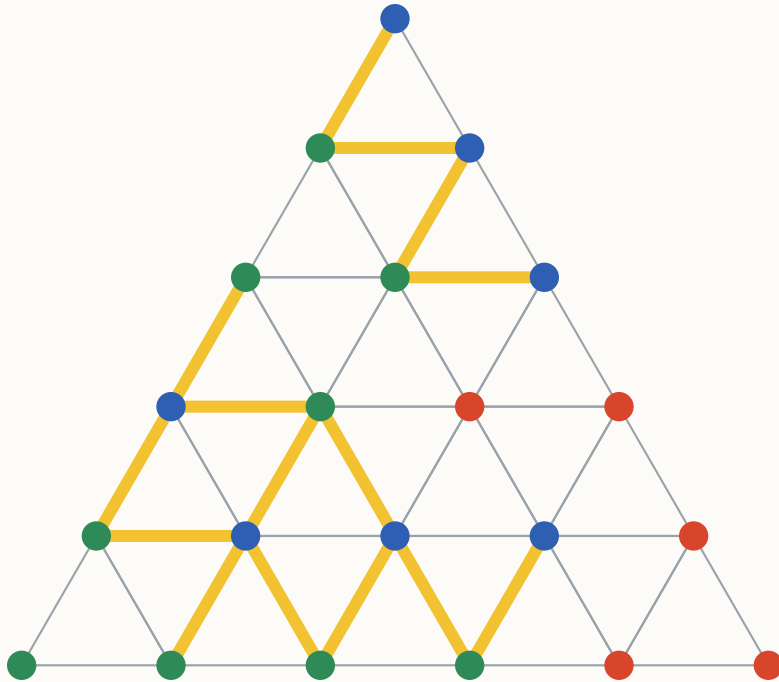


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Question. How many doors can a room have? Which rooms have exactly one?

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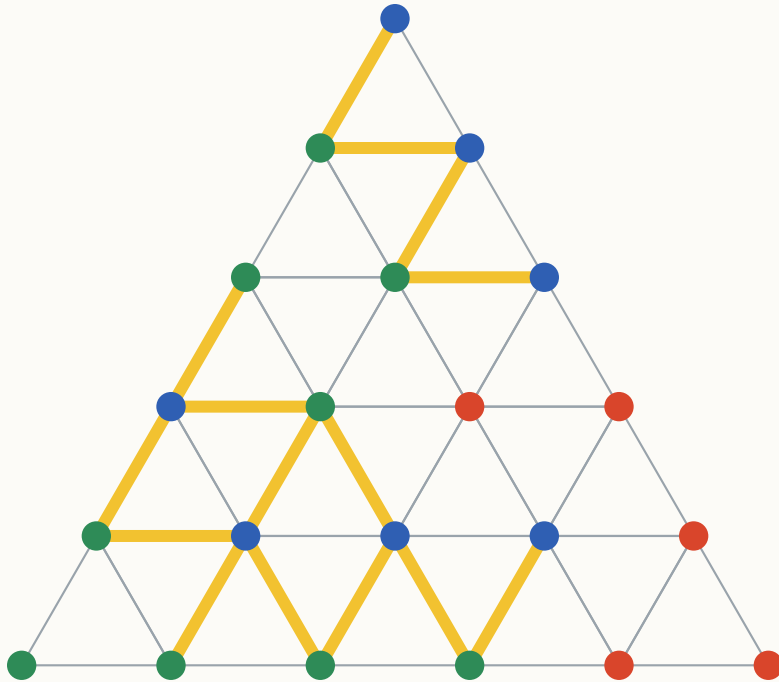


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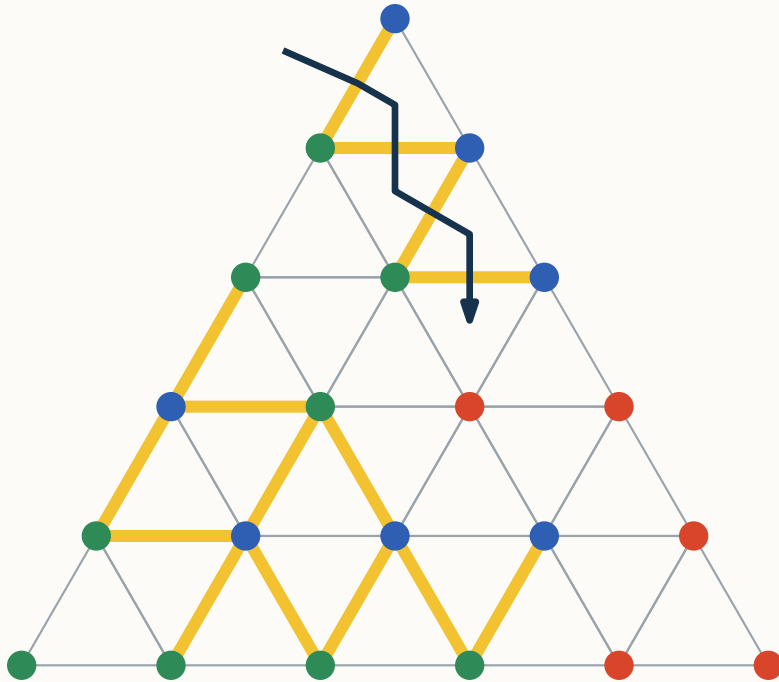
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Question. Enter through a boundary door. What can happen in the room?

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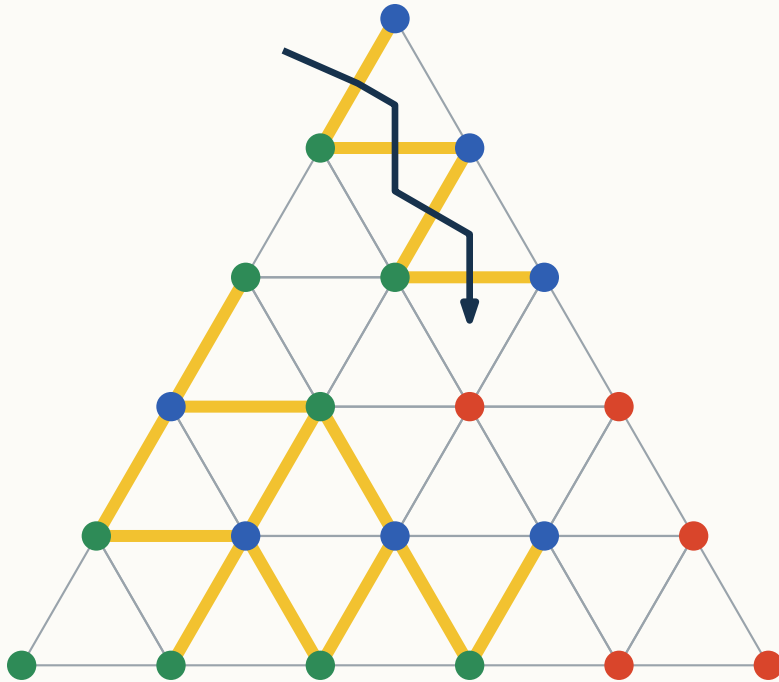
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Walk: 1 door $\Rightarrow$ stop, it is fully coloured; 2 doors $\Rightarrow$ leave by the other one, and repeat.

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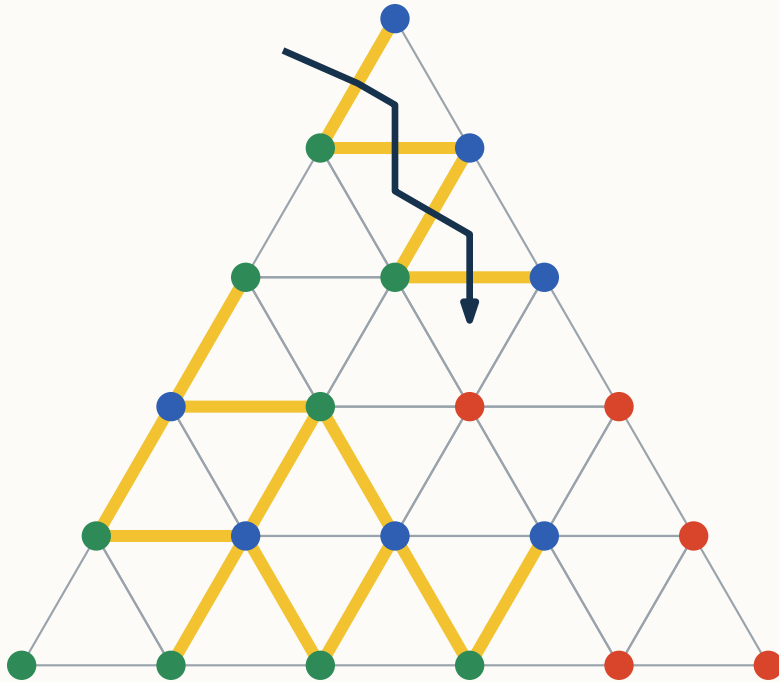
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Question. Can the walk visit a room twice? Does it end?

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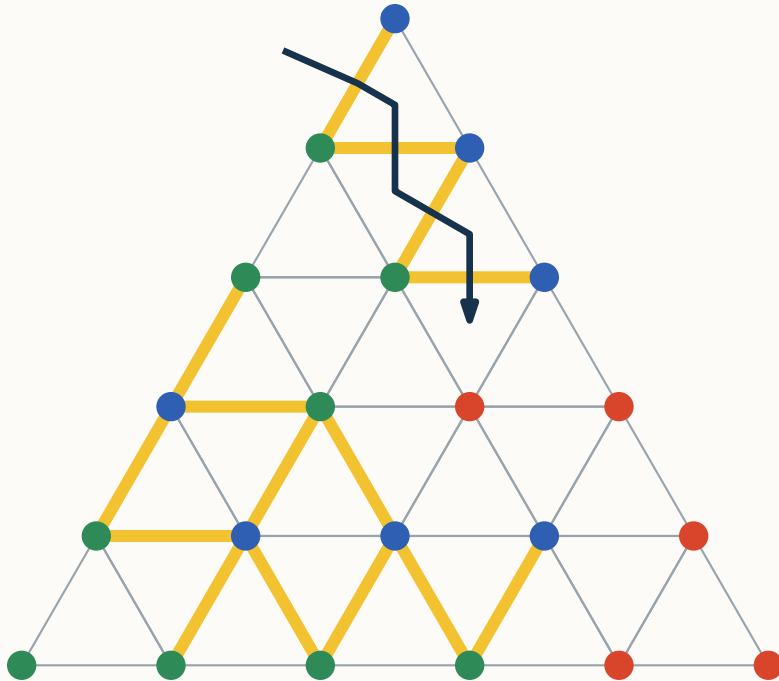
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No room twice: let R be the first room entered a second time. Both doors of R were used on its first visit, so we re-enter from a room visited before $\Rightarrow$ that room was revisited earlier. Contradiction. Finitely many rooms $\Rightarrow$ the walk ends.

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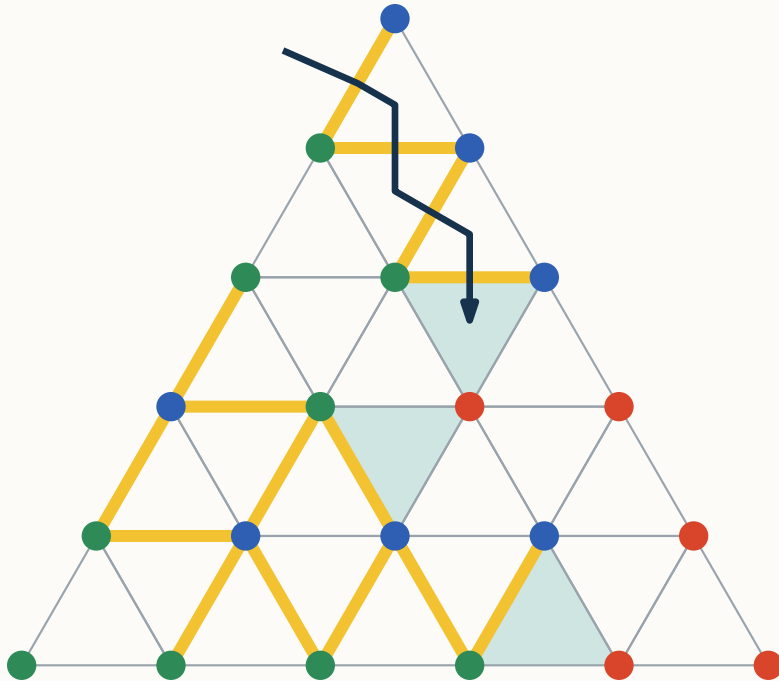
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Question. How many walks from the boundary end in a fully coloured room?

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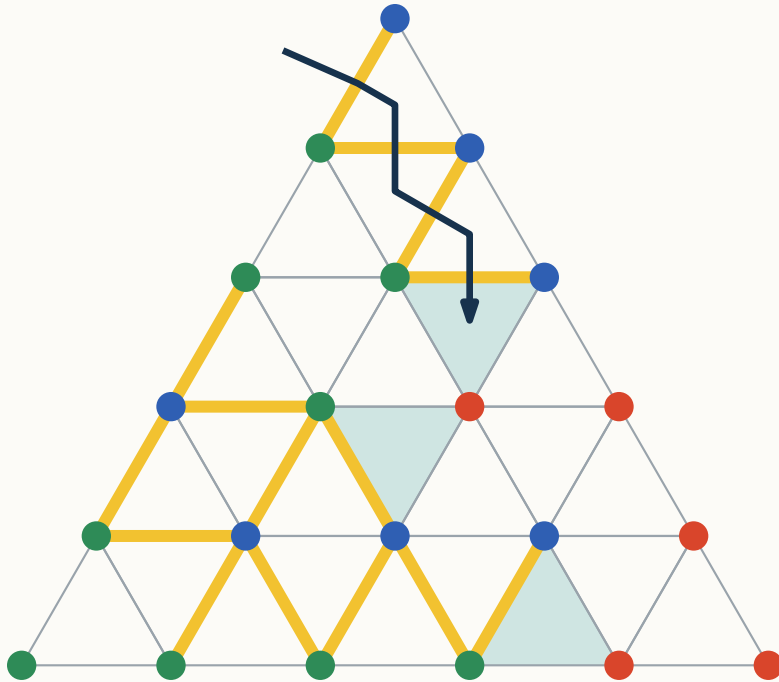
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No room twice: let R be the first room entered a second time. Both doors of R were used on its first visit, so we re-enter from a room visited before $\Rightarrow$ that room was revisited earlier. Contradiction. Finitely many rooms $\Rightarrow$ the walk ends.

Pairing: walks from boundary to boundary use up boundary doors in pairs; odd $\Rightarrow$ an **odd** number of walks end in a **fully coloured** room.

Proof: Walking Through Doors

House = big triangle, rooms = baby triangles, **door** = edge coloured $\{1, 2\}$



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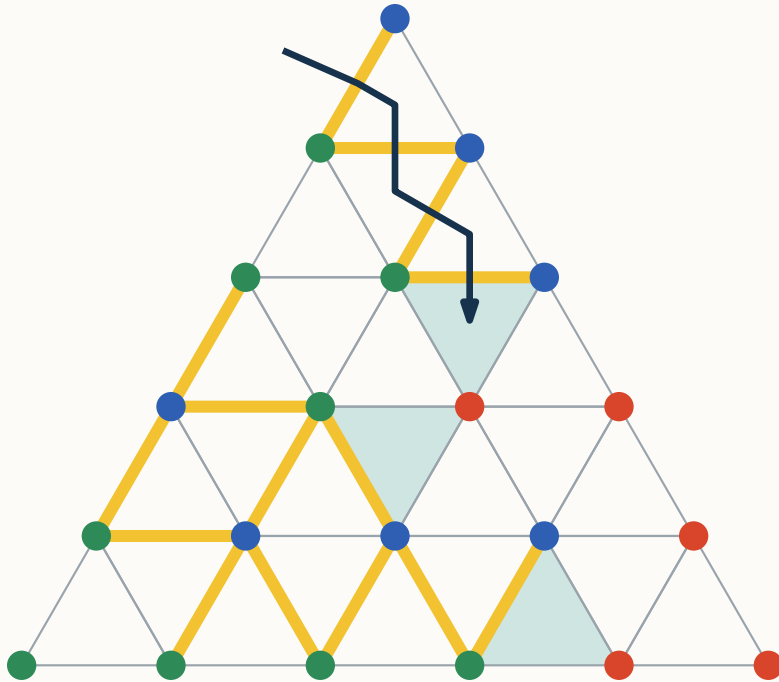
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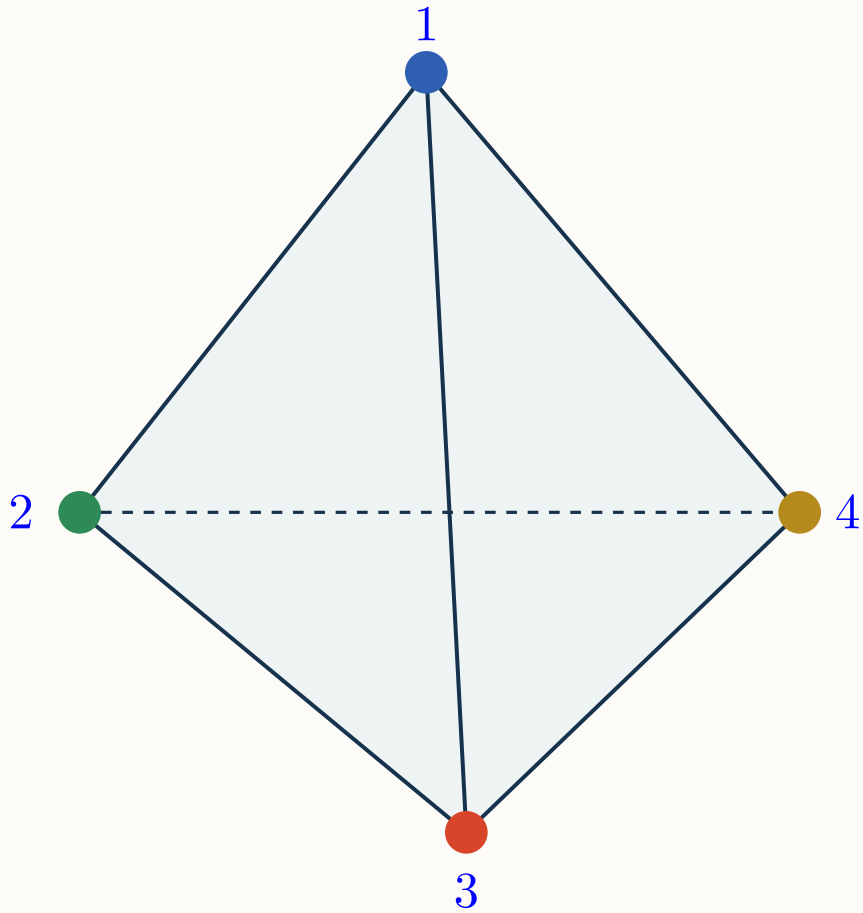
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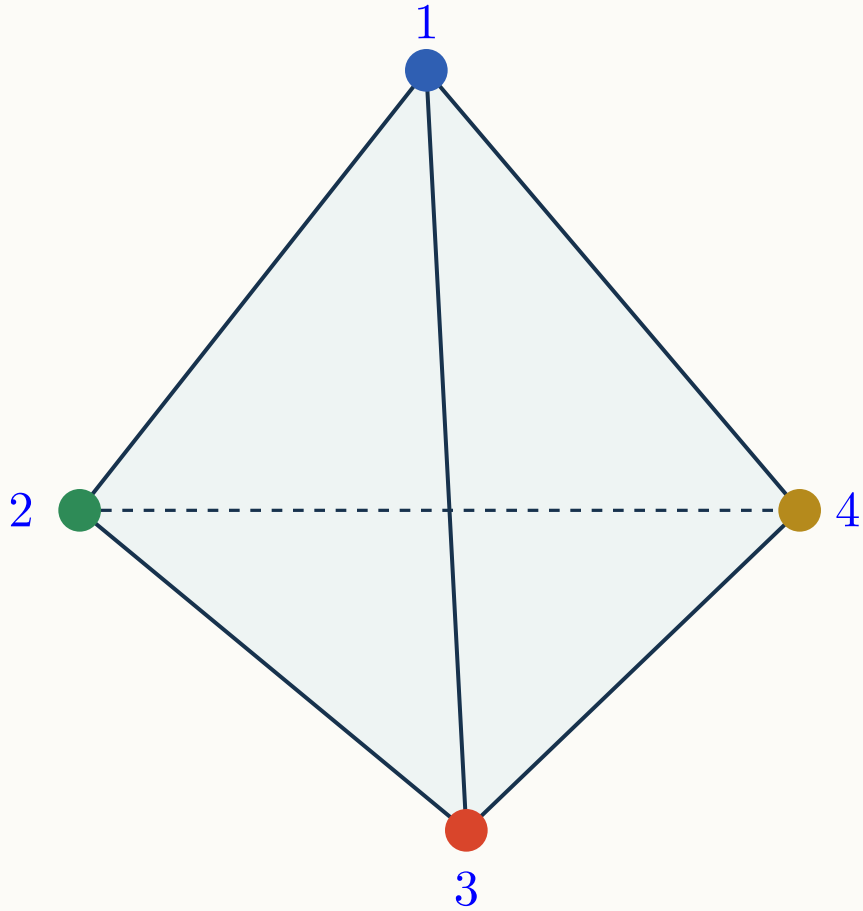
Parity: remaining fully coloured rooms are joined in pairs by walks $\Rightarrow$ **odd** in total.

Sperner in Any Dimension



Theorem (Sperner's lemma, n D). Triangulate an n -simplex with vertices $v_1, \dots, v_{n+1}$. Colour v_k with k , and every point on the face spanned by $\{v_k : k \in K\}$ with a colour from K . Then an **odd** number of small simplices use all $n + 1$ colours.

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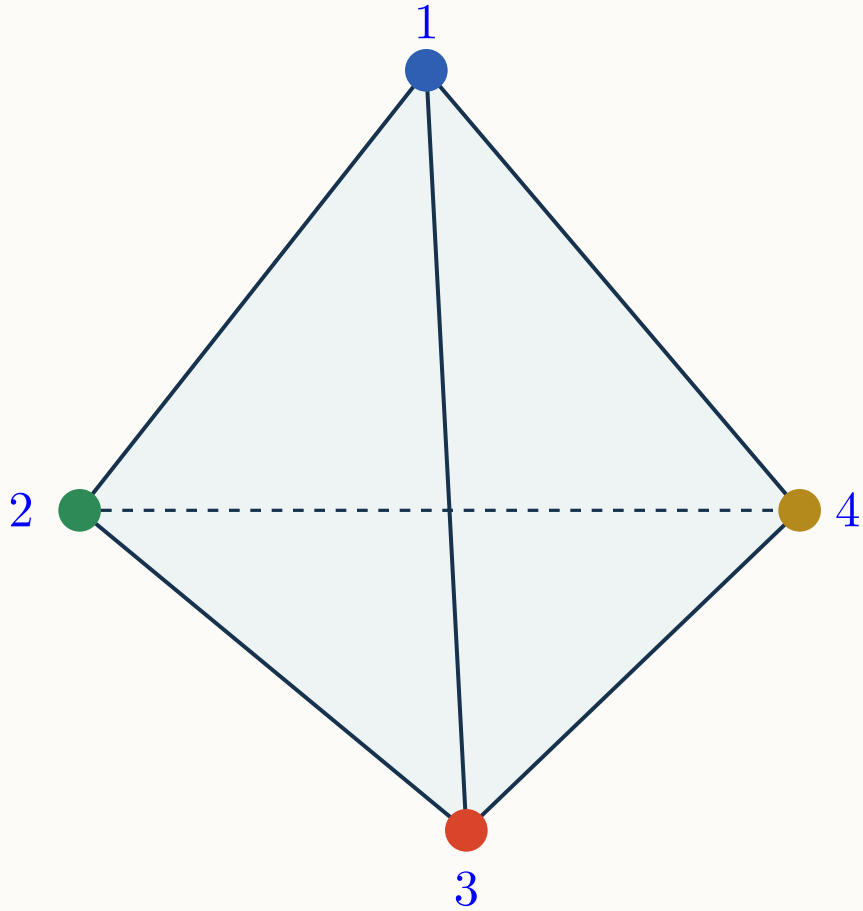


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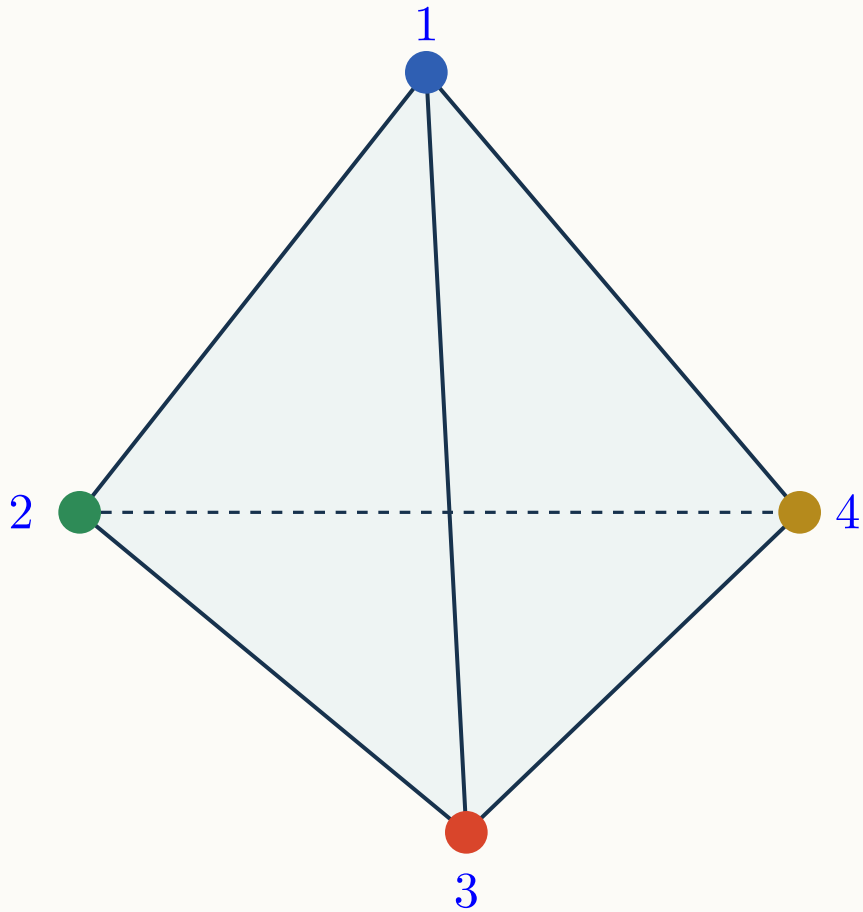


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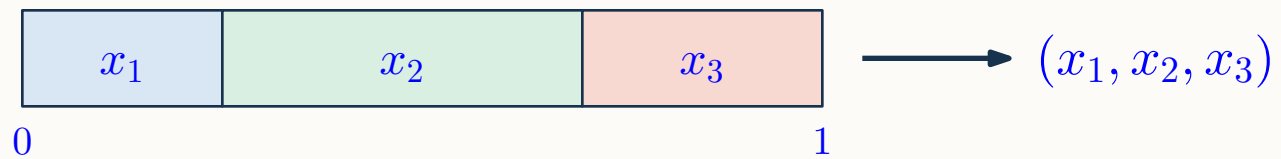


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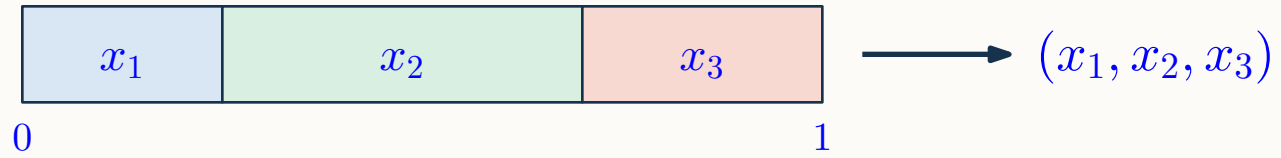
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- boundary doors lie on the face $v_1 \cdots v_n$, which carries an $(n - 1)$ D Sperner colouring $\Rightarrow$ odd by induction

Envy-free Division via Sperner: the Setup

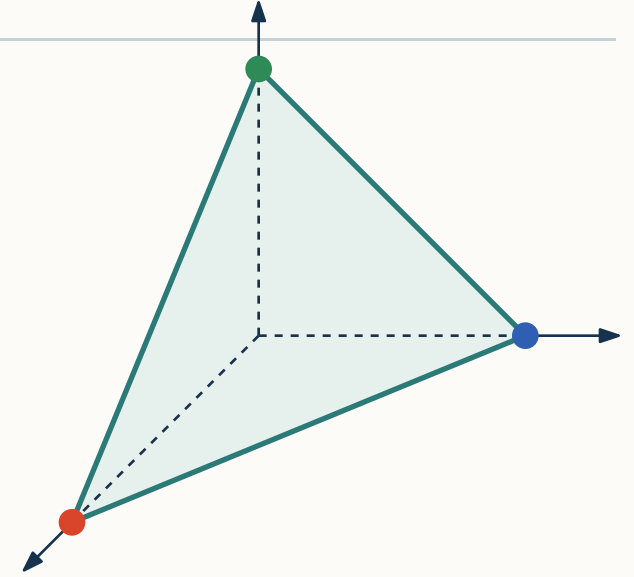


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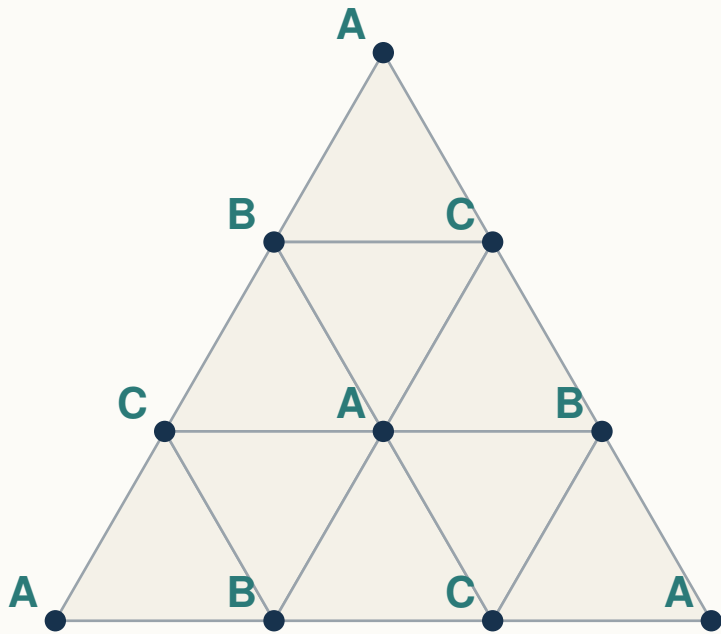


$$\text{Cuts} = \Delta_{n-1} = \{x \in \mathbb{R}^n : x \geq 0, x_1 + \cdots + x_n = 1\}$$

$n = 3$: a **triangle**

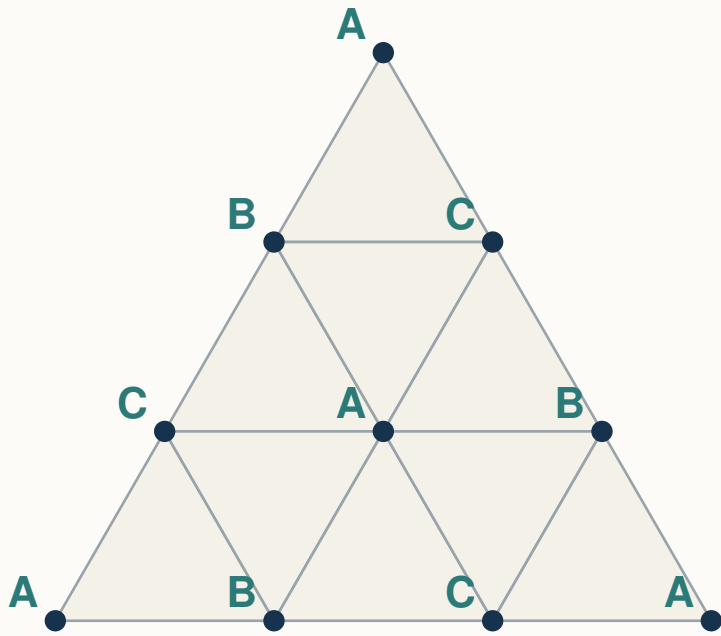


Owners and Colours

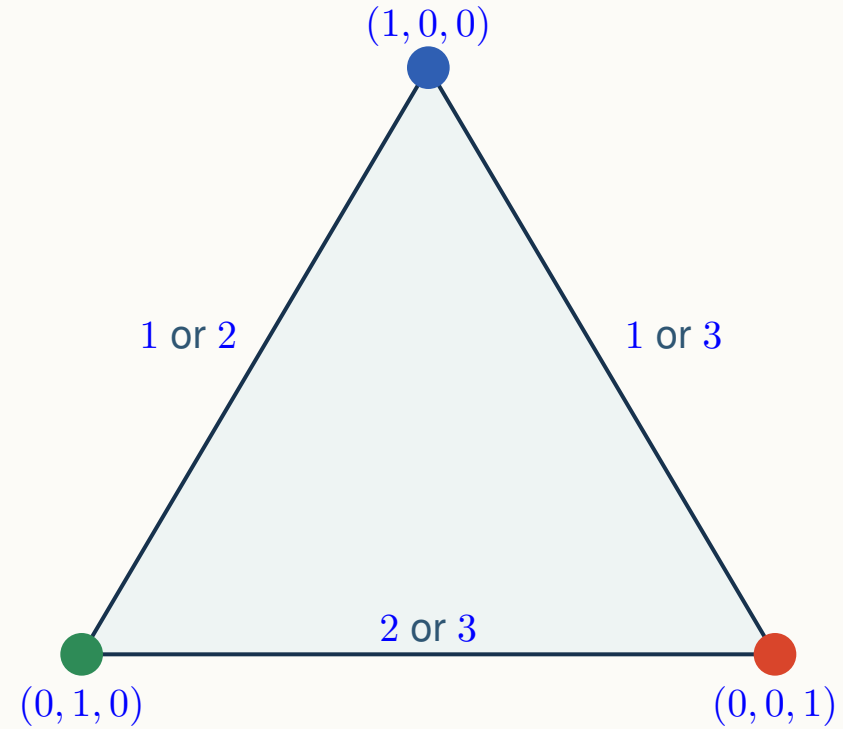


Owners: every baby triangle has one *A*, one *B*, one *C*

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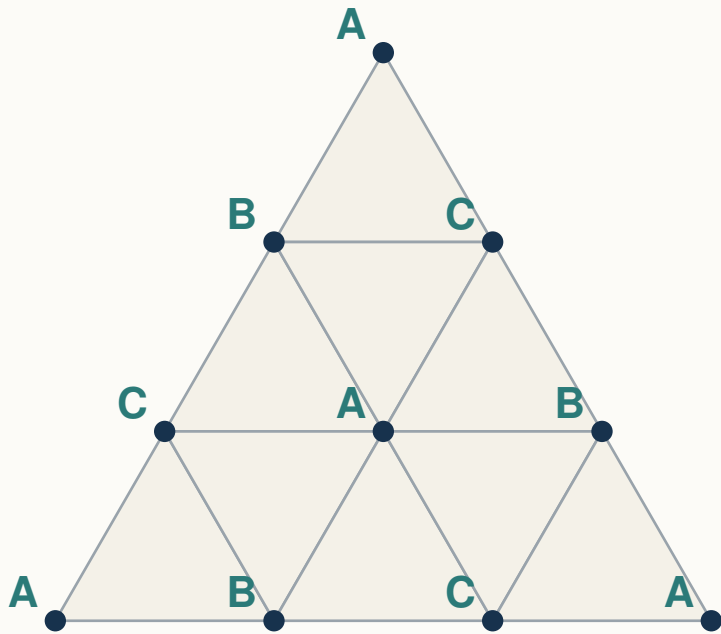


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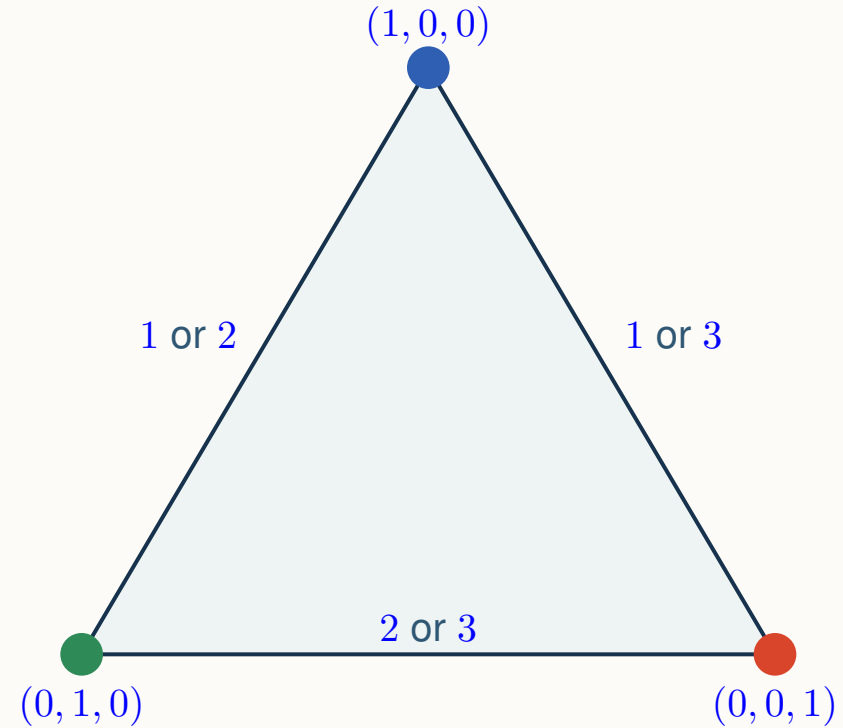


Colour of vertex x : a non-empty piece its owner prefers at cut x

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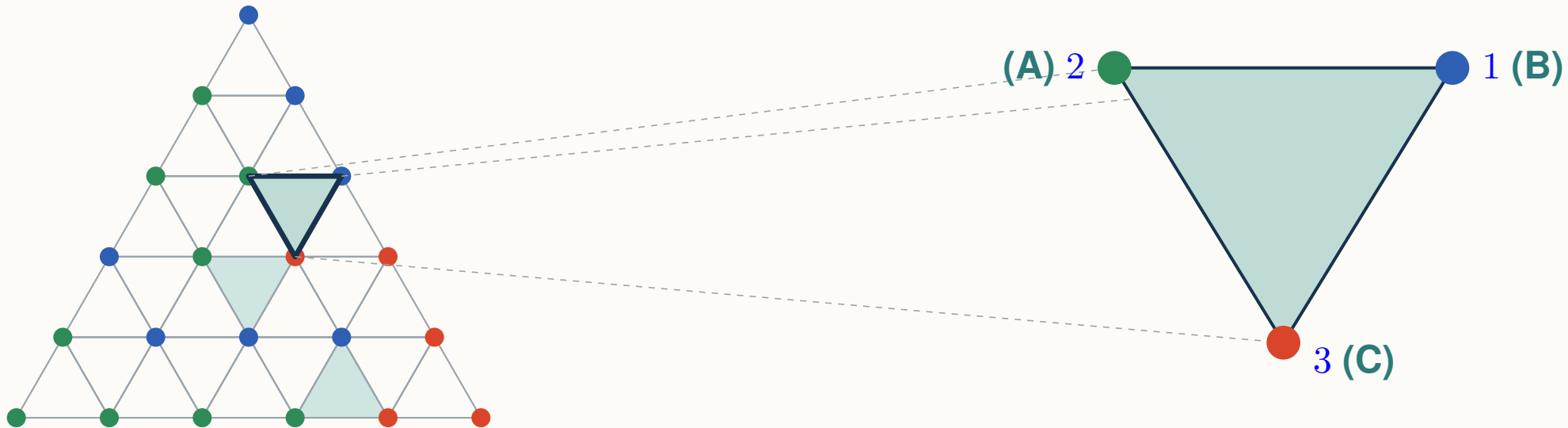


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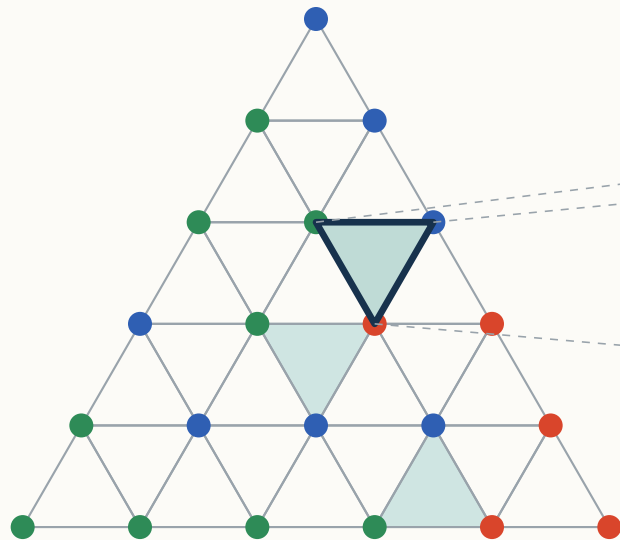
Claim. This is a Sperner colouring.

- At $(1, 0, 0)$, piece 1 is the whole cake and it is chosen (same for the other corners).
- On the side $x_3 = 0$, piece 3 is empty, so piece 1 or 2 is chosen (same for the other sides).

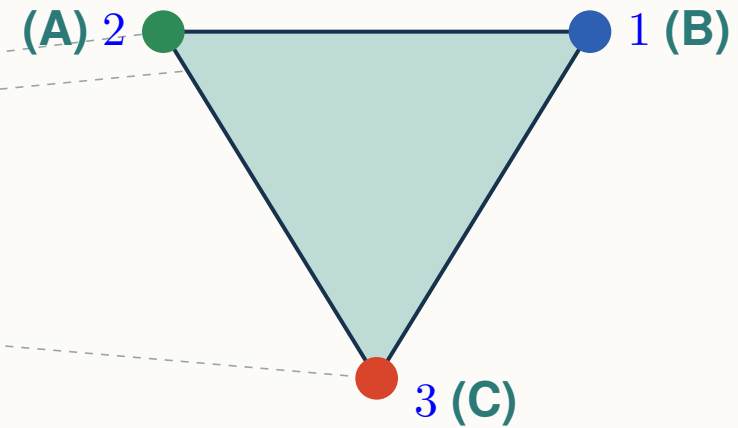
Three Nearby Cuts



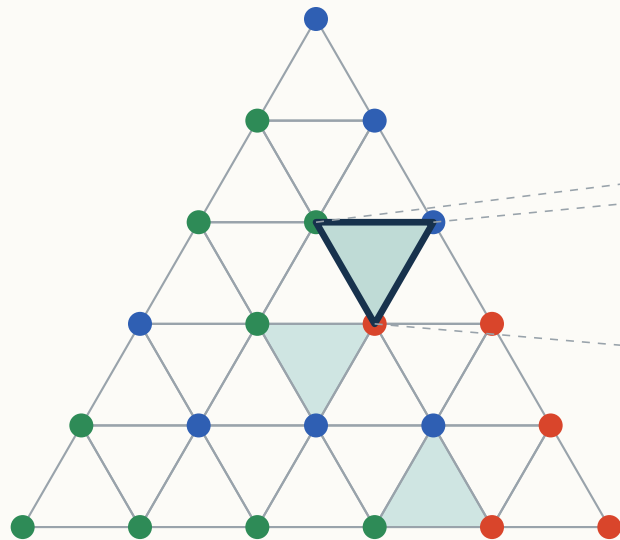
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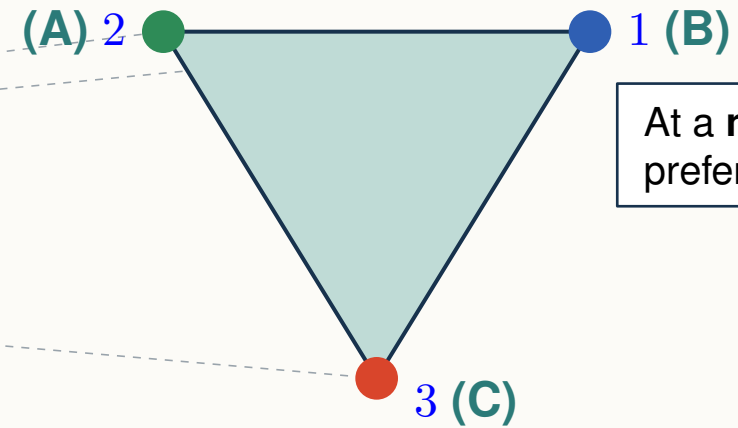
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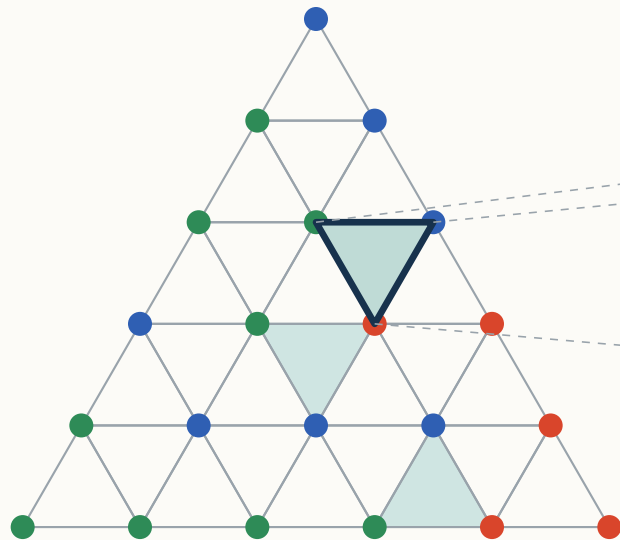


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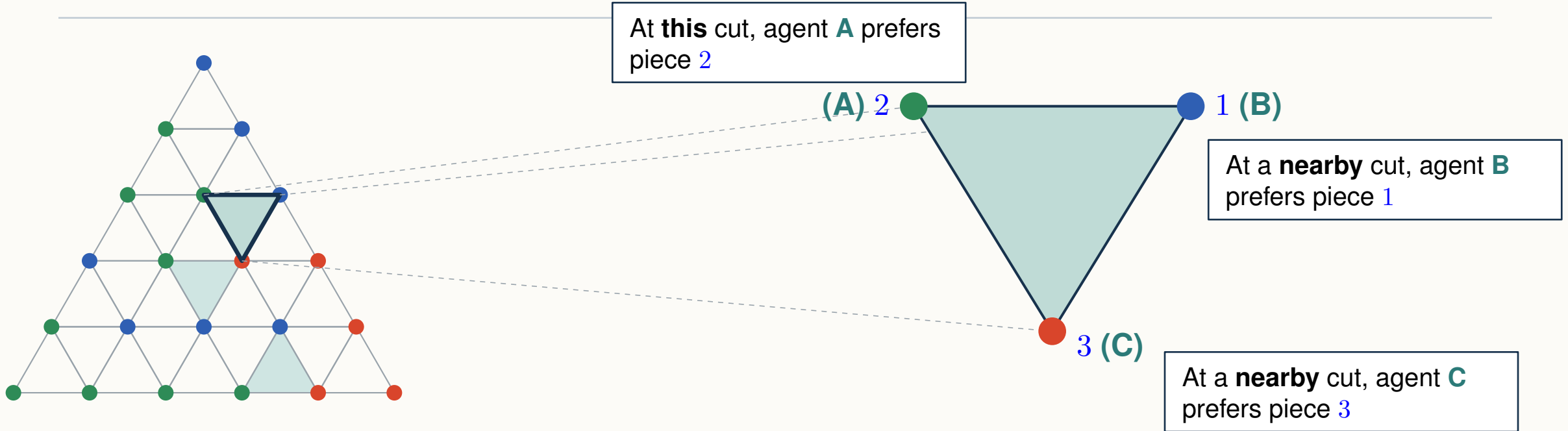
1 (B)

At a **nearby** cut, agent **B** prefers piece **1**

3 (C)

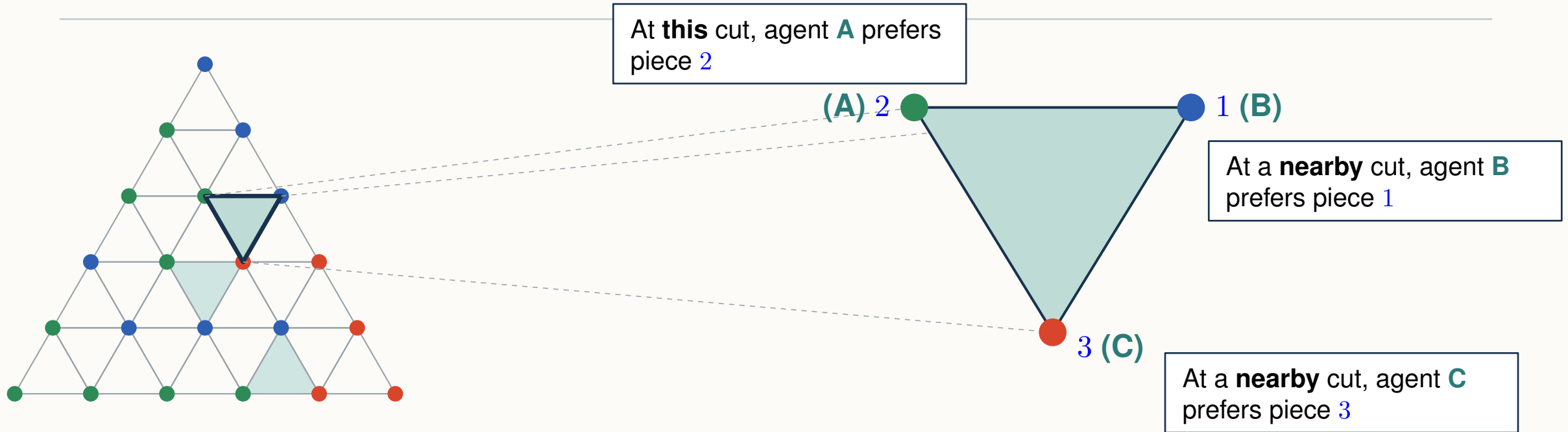
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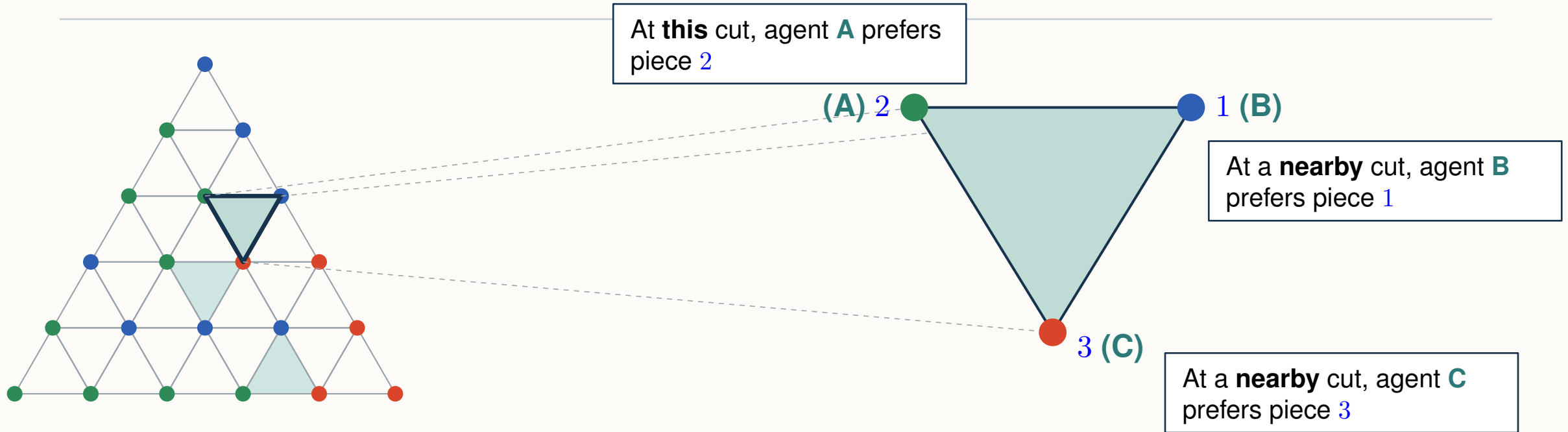
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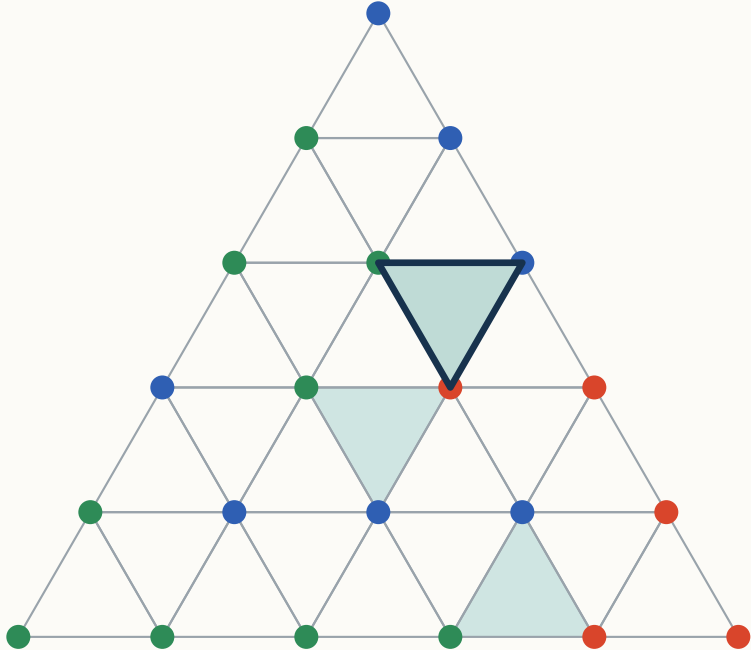


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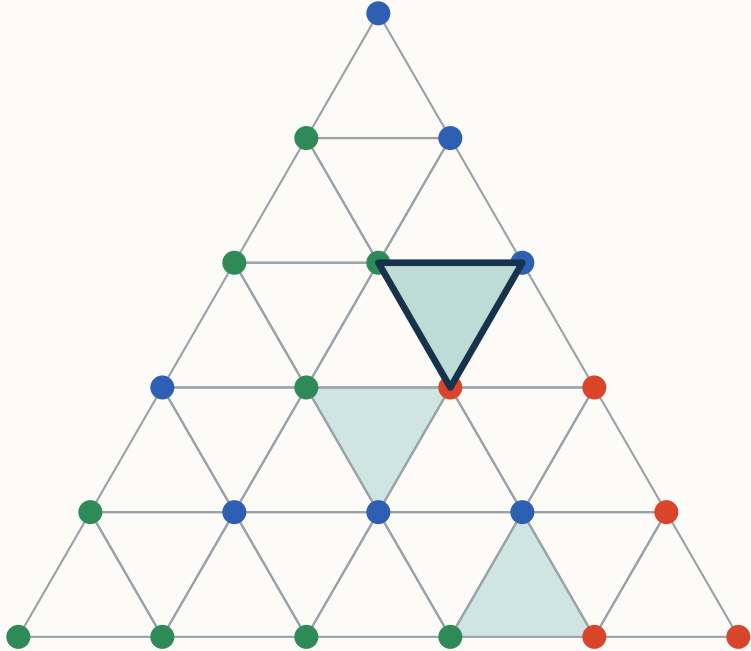
Sperner $\Rightarrow$ three nearby cuts, different favourites: **approximately envy-free contiguous division**

How to Get a Single Cut?



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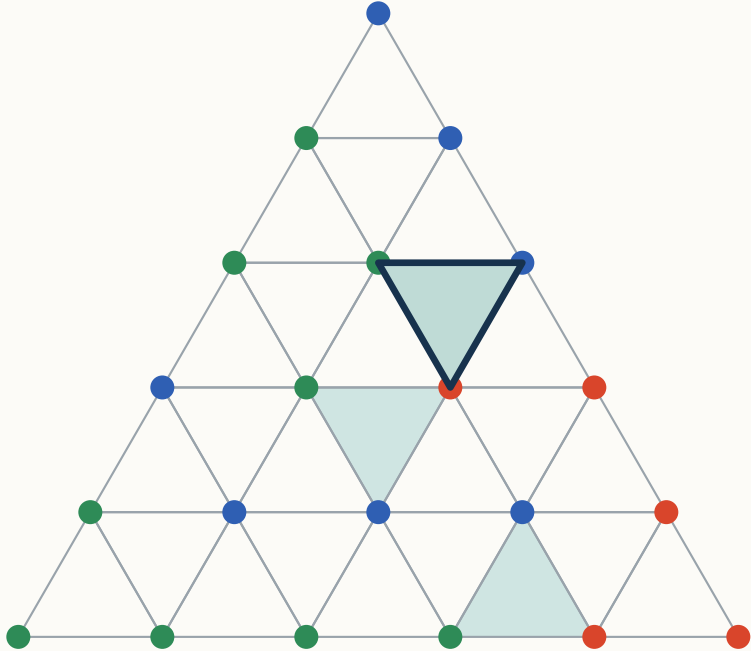
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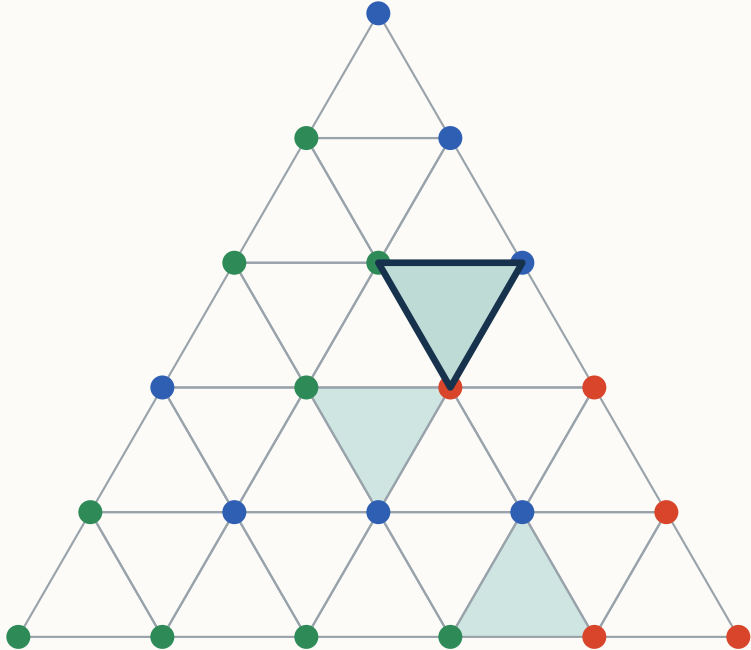


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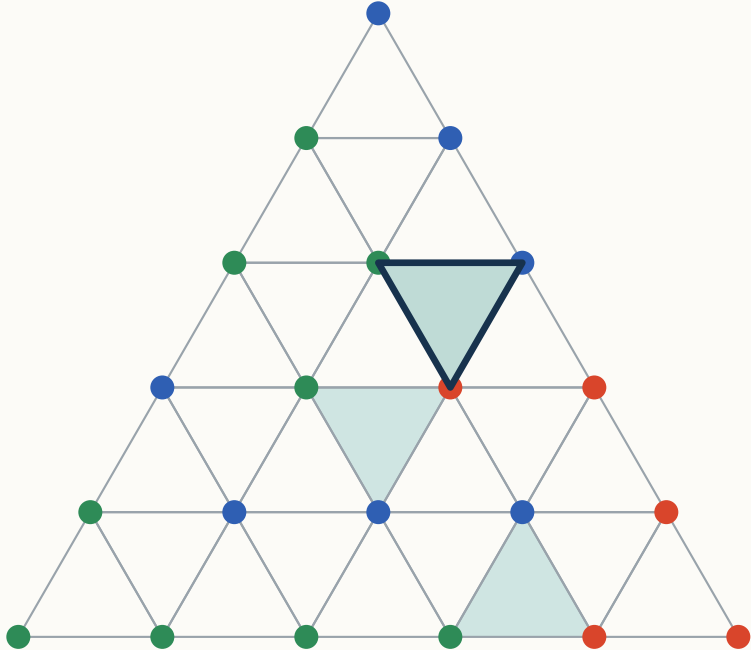
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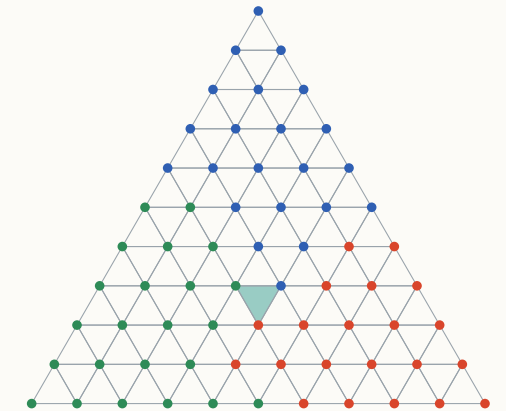
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We can do more: use **convergence**.

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A cut $X = (x_1, x_2, x_3)$. Consider a sequence of cuts $X^{(1)}, X^{(2)}, X^{(3)}, \dots$

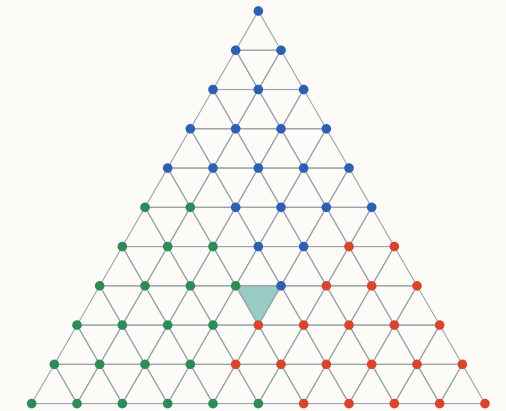


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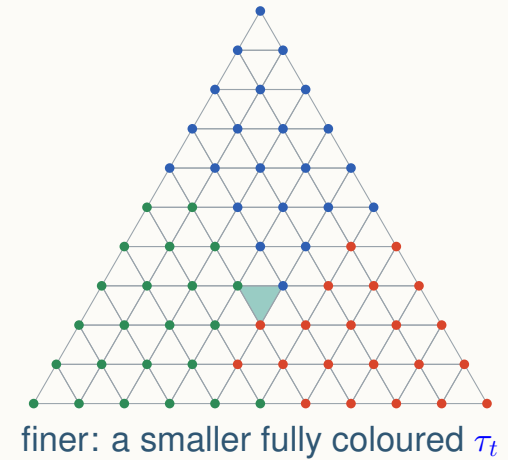
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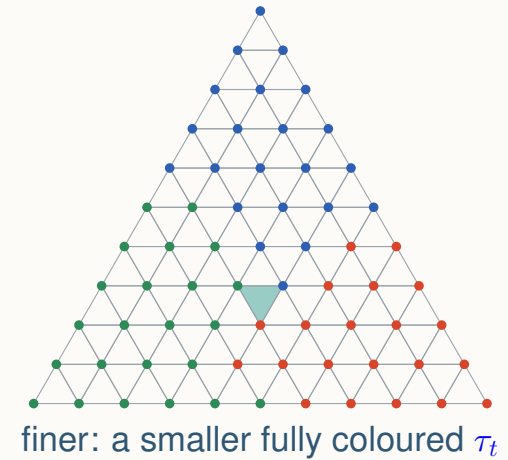
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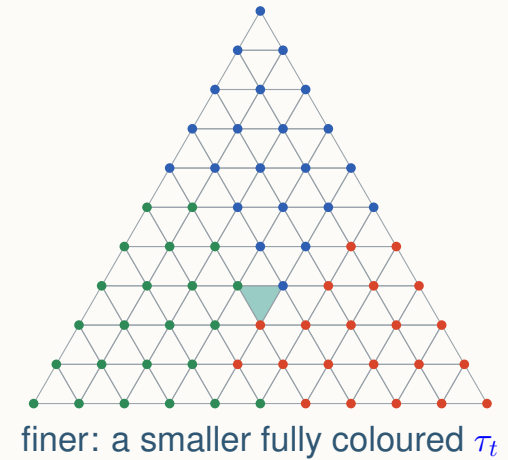
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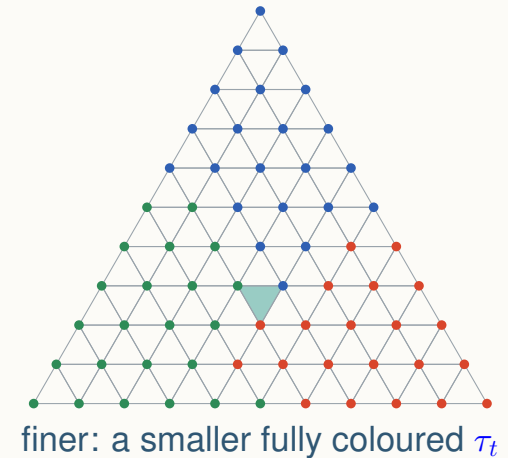
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$\Rightarrow$ **Exact envy-free contiguous division**



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- EF for n : bounded protocols exist, but a huge gap remains ($\Omega(n^2)$ vs. $n^{n^{n^n}}$)
- EF **and** contiguous: always exists (Sperner), but no finite protocol finds it
- The proof is existential; a fixed triangulation gives $\approx$ EF constructively

Exercises

Q 1 (Stromquist's moving-knife procedure, 1980). Valuations have a continuous, strictly positive density.

For 3 agents:

- A referee moves a *sword* continuously from 0 to 1; let x be its position.
- Each agent i holds a *knife* at $k_i \in [x, 1]$ with $v_i([x, k_i]) = v_i([k_i, 1])$. Let y be the *middle* knife and $L = [0, x]$, $M = [x, y]$, $R = [y, 1]$.
- An agent shouts “stop” as soon as they value L at least as much as both M and R .
- Cut at x and y . The shouter gets L ; of the other two, the one whose knife is further left gets M , the other gets R .

(a) Prove that the procedure terminates and outputs an envy-free contiguous allocation.

(b) Now the cake is a *chore*: agent i has a cost function c_i and wants a piece of *minimum* cost (EF: $c_i(A_i) \leq c_i(A_j)$). Design a moving-knife procedure with 2 cuts giving an envy-free contiguous allocation among 3 agents, and prove it correct.

Exercises

Q 2 (Sperner's lemma with orientation). Consider a Sperner-coloured triangulation of a triangle whose main vertices v_1, v_2, v_3 appear in counterclockwise order. Call a fully coloured baby triangle *positive* if its colours 1, 2, 3 appear in counterclockwise order, and *negative* otherwise. Prove that

$$\#\{\text{positive baby triangles}\} - \#\{\text{negative baby triangles}\} = 1.$$

In particular, this gives another proof that the number of fully coloured baby triangles is odd.

Hint: walk counterclockwise around the boundary of each baby triangle, and count the edges you traverse from colour 1 to colour 2 minus those you traverse from colour 2 to colour 1. What happens when you sum this over all baby triangles?