

Lecture 2: Envy-free Cake-cutting and Sperner's Lemma

19 October 2026

These notes are partly based on lecture notes by Masoud Seddighin and lecture slides by Nidhi Rath.

Outline of this session. Envy-free division for three agents via Selfridge–Conway, the query complexity of envy-freeness, Sperner's lemma, and the existence of envy-free contiguous divisions for any number of agents.

Recap

Recall the setting from Lecture 1: the cake is $C = [0, 1]$, the agents are $N = [n]$, and agent i has a valuation v_i that is normalized, divisible and non-negative. An allocation $A = (A_1, \dots, A_n)$ is

- **proportional** if $v_i(A_i) \geq \frac{1}{n}$ for every agent i (Steinhaus, 1948);
- **envy-free (EF)** if $v_i(A_i) \geq v_i(A_j)$ for every pair of agents i, j (Foley, 1967).

In non-wasteful allocations, EF implies proportionality, and the two notions coincide for $n = 2$. We also saw that proportional allocations can be computed in the Robertson–Webb model with $O(n \log n)$ queries (Even–Paz), and that this is tight (Edmonds & Pruhs).

In this lecture we turn to the stronger notion, envy-freeness. For two agents, *cut & choose* is envy-free and uses 2 queries. What about $n \geq 3$?

Envy-free Division for Three Agents

Example 1 (Selfridge–Conway, 1960). An envy-free division of a cake among 3 agents.

1. Agent 1 divides the cake into three equal parts X_1, X_2, X_3 : (2 queries)

$$v_1(X_1) = v_1(X_2) = v_1(X_3).$$

2. Without loss of generality, suppose agent 2 values the three pieces as (3 queries)

$$v_2(X_1) \geq v_2(X_2) \geq v_2(X_3).$$

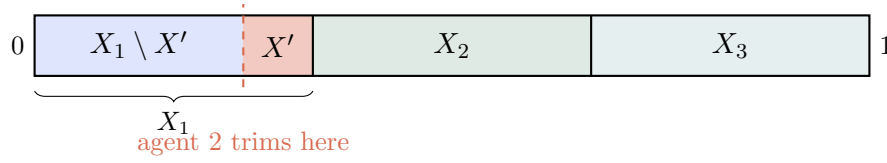
Agent 2 trims a piece X' off X_1 so that the remainder of X_1 is worth exactly as much as X_2 . The removed piece X' is called the *trimmings*:

$$v_2(X_1 \setminus X') = v_2(X_2) \geq v_2(X_3).$$

3. Agent 3 chooses the piece they like best among $X_1 \setminus X'$, X_2 and X_3 . (3 queries)
4. If agent 3 did not choose $X_1 \setminus X'$, agent 2 takes it; otherwise agent 2 takes X_2 . (0 queries)

5. Agent 1 takes the remaining piece.

(0 queries)



At this point, the whole cake except X' has been allocated, and this partial allocation is EF. Let $T \in \{2, 3\}$ be the agent who took the trimmed piece $X_1 \setminus X'$, and let $\bar{T}$ be the other agent in $\{2, 3\}$.

Dividing the trimmings X' :

6. Agent $\bar{T}$ divides X' into three pieces of equal value.

(2 queries)

7. Agents T , 1 and $\bar{T}$, in this order, each choose their preferred remaining piece. (4 queries)

It remains to show that no agent envies another.

Proof of envy-freeness. Each agent's final share is the union of a piece from *stage 1* (steps 1–5) and a piece from *stage 2* (steps 6–7). Since valuations are additive, it suffices to show that, for every pair i, j , agent i does not envy j in either stage, except in one case that we handle directly.

Stage 1.

- Agent 3 chooses first, so agent 3 envies no one.
- Agent 3 takes at most one of $X_1 \setminus X'$ and X_2 , so agent 2 always receives one of them. Both are agent 2's favourites, since $v_2(X_1 \setminus X') = v_2(X_2) \geq v_2(X_3)$. Hence agent 2 envies no one.
- By step 4, $X_1 \setminus X'$ is always taken by agent 2 or agent 3, so agent 1 receives X_2 or X_3 , worth $\frac{1}{3}$ to them. The other pieces are worth $v_1(X_2) = v_1(X_3) = \frac{1}{3}$ and $v_1(X_1 \setminus X') \leq v_1(X_1) = \frac{1}{3}$. Hence agent 1 envies no one.

Stage 2.

- Agent T chooses first, so T envies no one.
- Agent $\bar{T}$ cut X' into three pieces of equal value to them, so $\bar{T}$ envies no one, whichever piece is left.
- Agent 1 chooses before $\bar{T}$, so agent 1 does not envy $\bar{T}$. Agent 1 *may* envy T in stage 2, but not overall: T 's total share is $(X_1 \setminus X') \cup Y$ for some $Y \subseteq X'$, so

$$v_1((X_1 \setminus X') \cup Y) \leq v_1(X_1) = \frac{1}{3} \leq v_1(\text{agent 1's share}).$$

Summing the two stages, no agent envies another. $\square$

Properties:

Proportional: ✓ Envy-free: ✓ Equitable: ✗
 Contiguous: ✗ Pareto optimal: ✗ Non-wasteful: ✓

Number of queries. The procedure can be implemented in the Robertson–Webb model using $2 + 3 + 3 + 2 + 4 = 14$ queries. It makes up to 5 cuts: 2 by agent 1, 1 trim, and 2 to divide the trimmings, so the pieces are in general *not* contiguous.

Envy-free Division for n Agents: Query Complexity

Selfridge–Conway was generalised to n agents by Brams & Taylor (1995) and Robertson & Webb (1997). However, these protocols are *finite but unbounded*: there is no upper bound on the number of queries (or cuts) that depends only on n .

Theorem 1 (Procaccia 2009). *Every protocol for envy-free cake division requires $\Omega(n^2)$ queries.*

Theorem 2 (Aziz & Mackenzie 2016). *There is a protocol that computes an envy-free division among n agents using $n^{n^{n^n}}$ queries.*

(FOCS'16: *A discrete and bounded envy-free cake cutting protocol for any number of agents.*)
The number of cuts is of roughly the same order.

The following table summarises what we know in the Robertson–Webb model.

	Proportional	Envy-free
$n = 2$	2 (cut & choose)	2 (cut & choose)
$n = 3$		14 (Selfridge–Conway)
Upper bound	$O(n^2)$ (Dubins–Spanier)	finite, unbounded (Brams–Taylor)
	$O(n \log n)$ (Even–Paz)	$O(n^{n^{n^n}})$ (Aziz–Mackenzie)
Lower bound	$\Omega(n \log n)$ (Edmonds–Pruhs)	$\Omega(n^2)$ (Procaccia)

Closing the huge gap for envy-freeness is a major open problem.

What if we insist on *contiguous* pieces? Stromquist (2008) showed that for $n \geq 3$ *no* finite Robertson–Webb protocol can guarantee an envy-free allocation with contiguous pieces. Nevertheless, they always *exist*:

Theorem 3 (Stromquist 1980; Su 1999). *For every n , there exists an envy-free and contiguous allocation of the cake.*

We prove this theorem below, following the elegant argument of Forest Simmons, popularised by Francis Su (1999). The main tool is a combinatorial lemma that, on the face of it, has nothing to do with cake division.

Sperner's Lemma

The 1-dimensional case

A line segment is subdivided by a number of points. The left endpoint is coloured 1 and the right endpoint 2. Every intermediate point is coloured 1 or 2 arbitrarily. Then

- (a) there is at least one multicoloured segment, and
- (b) the number of multicoloured segments is odd.

Proof. Walking from left to right, we start at colour 1 and end at colour 2, so the colour must switch an odd number of times. $\square$

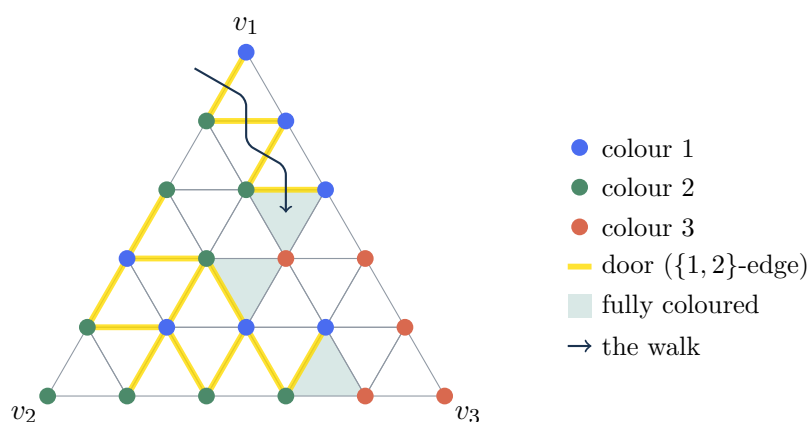
The 2-dimensional case

Definition 1. Let T be a triangle with vertices v_1, v_2, v_3 , *triangulated* into small (“baby”) triangles, so that every edge of every baby triangle joins two of the points. A **Sperner colouring** assigns a colour from $\{1, 2, 3\}$ to every point such that

- the main vertex v_k gets colour k ;
- points on side v_1v_2 are coloured 1 or 2; on side v_1v_3 , 1 or 3; on side v_2v_3 , 2 or 3;
- interior points are coloured 1, 2 or 3 arbitrarily.

A baby triangle is **fully coloured** if its vertices have all three colours.

Theorem 4 (Sperner’s lemma, 2D). *Every Sperner-coloured triangulation contains a fully coloured baby triangle; in fact, it contains an odd number of them.*



We give two proofs. The first is a constructive “walk”, the second a short counting argument.

Proof 1: walking through doors. Think of the big triangle as a *house* and of the baby triangles as *rooms*. An edge whose endpoints have colours 1 and 2 is a *door*.

Observation 1: the number of doors on the boundary of the house is odd. Doors on the boundary can only lie on side v_1v_2 , since the other two sides do not use both colours 1 and 2. Side v_1v_2 is a 1-dimensional Sperner colouring, so it has an odd number of doors by the 1-dimensional case.

Observation 2: every room has 0, 1 or 2 doors, and it has exactly 1 door if and only if it is fully coloured. If a room is coloured $\{1, 2, 3\}$, exactly one of its edges is a $\{1, 2\}$ -edge. If it is coloured $\{1, 1, 2\}$ or $\{1, 2, 2\}$, it has two doors. Otherwise it has none.

Now enter the house through a boundary door. The room we enter has 1 or 2 doors. If it has one door, it is fully coloured and we are done. If it has two, we leave through the other door (never through the door we entered by) into a new room, and keep walking. *The walk never visits a room twice.* Suppose it does, and let R be the first room entered for the second time, coming from a room Q through a door d of R . On the first visit to R both of its doors were used, so d was already crossed before, between R and Q . If this happened in the step just before, i.e. the walk went $R \rightarrow Q \rightarrow R$, then we left Q through the door we entered by, which the rule forbids. Otherwise Q was visited before, so Q is visited a second time before R is, contradicting the choice of R . Since there are finitely many rooms, the walk ends, either in a fully coloured room or by leaving the house through another boundary door.

Walks that leave the house pair up boundary doors: the walk from one boundary door exits through another, and walking backwards gives the walk between the same two doors. Since the number of boundary doors is odd, an odd number of them are not paired up, and the walks starting there end in distinct fully coloured rooms. *Parity.* A walk can also start in a fully

coloured room, through its only door. If it ends at a boundary door, it is one of the walks above walked backwards, so its room was already counted. Otherwise it ends in another fully coloured room, and the two rooms form a pair. Hence the number of fully coloured rooms is odd. $\square$

Proof 2: counting. For an edge e , let $\ell(e) = 0$ if its endpoints have the same colour and $\ell(e) = 1$ otherwise. For a baby triangle τ , let $s(\tau) = \sum_{e \in \tau} \ell(e)$ be the sum over its three edges. Then $s(\tau) \in \{0, 2, 3\}$: it is 0 if τ uses one colour, 2 if it uses two, and 3 exactly when τ is fully coloured. Hence the number of fully coloured triangles has the same parity as $\sum_{\tau} s(\tau)$, and it suffices to show that this sum is odd.

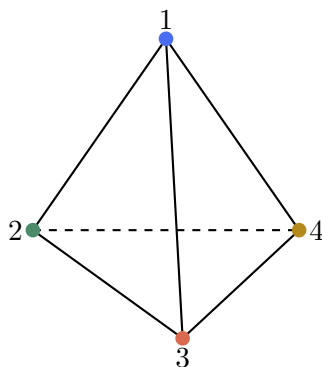
Every interior edge belongs to exactly two baby triangles and every boundary edge to exactly one, so

$$\sum_{\tau} s(\tau) = 2 \underbrace{\sum_{e \text{ interior}} \ell(e)}_{\text{even}} + \underbrace{\sum_{e \subseteq v_1 v_2} \ell(e)}_{\text{odd}} + \underbrace{\sum_{e \subseteq v_1 v_3} \ell(e)}_{\text{odd}} + \underbrace{\sum_{e \subseteq v_2 v_3} \ell(e)}_{\text{odd}}.$$

Each side is a 1-dimensional Sperner colouring, so each boundary sum counts an odd number of multicoloured segments. Hence the total is even + odd + odd + odd, which is odd. $\square$

Higher dimensions

The same holds in every dimension, e.g. for a tetrahedron:



Theorem 5 (Sperner's lemma, nD). *Let an n -dimensional simplex with vertices $v_1, \dots, v_{n+1}$ be triangulated into small simplices. Colour the points so that v_k gets colour k , and every point on a face spanned by $\{v_k : k \in K\}$ gets a colour from K . Then there is an odd number of fully coloured small simplices (using all $n + 1$ colours), and in particular at least one.*

Both proofs generalise by induction on the dimension: in the walking proof, the doors are the $(n - 1)$ -dimensional faces coloured with exactly $\{1, \dots, n\}$, and the boundary doors lie on the face spanned by $v_1, \dots, v_n$, which carries an $(n - 1)$ -dimensional Sperner colouring.

Envy-free Cake Division via Sperner's Lemma

The space of cuts is a simplex

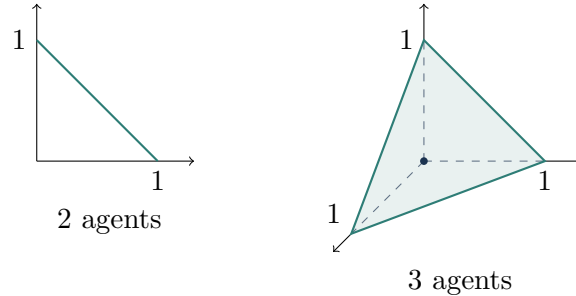
We first map every division of the cake into n contiguous pieces, read from left to right, to a point:

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n \\ | \quad | \quad | \quad | \quad | \quad | \\ 0 \quad \quad \quad \quad \quad \quad \quad 1 \end{array} \longrightarrow (x_1, x_2, x_3, \dots, x_n)$$

Here x_k is the length of the k -th piece. We call such a point a *cut*. Every point corresponds to a division and vice versa. The set of all cuts is

$$\Delta_{n-1} = \left\{ x \in \mathbb{R}^n : x_i \geq 0, x_1 + x_2 + \cdots + x_n = 1 \right\},$$

which, for n agents, is an $(n-1)$ -dimensional simplex. For $n=3$ it is a triangle.



Preferences of the agents

We present the proof for 3 agents A, B, C ; the same argument works for n agents. Given a cut x , agent i *prefers* piece k if piece k maximises v_i among the pieces of x , i.e. $v_i(\text{piece } k) \geq v_i(\text{piece } j)$ for all j . Preferences are weak: an agent may prefer several pieces (ties). We record two facts.

1. **A preferred piece always exists, and it is never empty.** The pieces of x cover the cake, so their values sum to $v_i(C) = 1$. Hence the best piece is worth at least $\frac{1}{n} > 0$, while an empty piece is worth 0.
2. **Preferences are closed.** If $x^{(1)}, x^{(2)}, \dots$ is a sequence of cuts converging to x^* , and agent i prefers piece k at every $x^{(t)}$, then agent i also prefers piece k at x^* .

The name comes from the fact that, for fixed i and k , the set

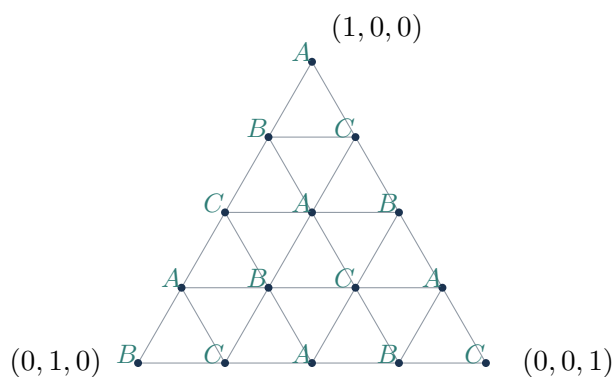
$$S_{ik} = \{x \in \Delta_{n-1} : \text{agent } i \text{ prefers piece } k \text{ at } x\}$$

is a closed set. This holds for our valuations: by divisibility, v_i has no atoms, so $v_i([a, b])$ is continuous in a and b ; hence the value of the k -th piece depends continuously on the cut x , and the weak inequalities $v_i(\text{piece } k) \geq v_i(\text{piece } j)$ are preserved in the limit. (With *strict* preferences this would fail: two pieces whose values approach each other can be strictly ordered all along a sequence but tied in the limit.)

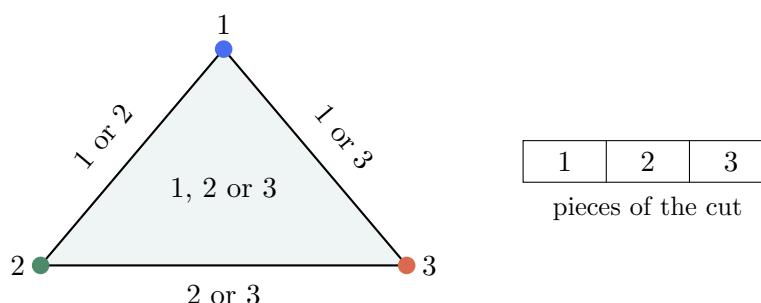
Remark (Su's general setting). Apart from these two facts, the argument below does not use the valuations at all. Su (1999) proves the existence of an envy-free division with contiguous pieces for arbitrary preferences over the pieces of a cut (which may even depend on the whole cut), assuming that (i) at every cut each agent prefers at least one piece, (ii) agents are *hungry*, i.e. never prefer an empty piece to a non-empty one, and (iii) preferences are closed. For our valuations, (i)–(iii) hold automatically by the two facts above.

Ownership labelling and Sperner colouring

Triangulate the simplex of cuts, and assign every vertex an *owner* from $\{A, B, C\}$ so that every baby triangle has one vertex owned by each of A, B, C . (Such an ownership labelling always exists and is easy to construct. For the regular triangulation below, label the vertex in row r and position k by $(r+k) \bmod 3$.)



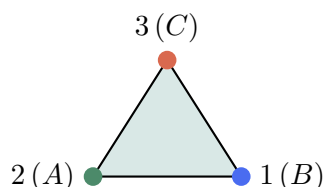
Now we construct a colouring. At every vertex x , we ask its owner which piece they prefer in the cut x , and colour the vertex with that piece's number.



Claim. This is a Sperner colouring.

- At the vertex $(1, 0, 0)$, piece 1 is the whole cake and the other pieces are empty. Preferred pieces are never empty, so piece 1 is chosen. Similarly, $(0, 1, 0)$ gets colour 2 and $(0, 0, 1)$ gets colour 3.
- On the side where $x_3 = 0$, piece 3 is empty, so it is never preferred, and piece 1 or 2 is chosen. The other two sides are similar.
- In the interior, any piece may be chosen.

By Sperner's lemma, there is a fully coloured baby triangle. Its three vertices are owned by A , B and C , and are coloured 1, 2 and 3. So, for example, at one of its corners agent A prefers piece 2, at a *nearby* cut agent B prefers piece 1, and at another nearby cut agent C prefers piece 3:



If all three agents' choices were at the *same* cut, this would be an envy-free division. Sperner's lemma gives us three *nearby* cuts at which different agents prefer different pieces; this is an "approximately" envy-free contiguous division. But these are three different cuts, not one cut.

How to get a single cut? A first idea is to zoom in: triangulate the fully coloured baby triangle again and apply Sperner's lemma inside it. This does not work. The colouring of the small triangle is *not* a Sperner colouring: its corners have three different colours, but the points on its sides may carry any colour, since nothing forces the owners' preferences there. Instead, we make the triangulation of the *whole* simplex finer and finer, and pass to a limit.

From approximate to exact envy-freeness

We use the following classical theorem from analysis.

Theorem 6 (Bolzano–Weierstrass). *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. In particular, if $S \subseteq \mathbb{R}^n$ is closed and bounded (e.g. the simplex Δ_{n-1}), then every sequence in S has a subsequence that converges to a point of S .*

We repeat the construction with finer and finer triangulations, whose baby triangles have diameter tending to 0. For each triangulation $t = 1, 2, \dots$, Sperner's lemma gives a fully coloured baby triangle τ_t .

- The triangles τ_t need not converge. But the simplex is bounded, so by the Bolzano–Weierstrass theorem a subsequence of their corners converges to a single cut x^* ; all three corners converge to x^* , since the diameters tend to 0.
- In each τ_t , the colouring defines a bijection between the agents $\{A, B, C\}$ and the pieces $\{1, 2, 3\}$. There are only $3! = 6$ such bijections, so one fixed bijection π occurs infinitely often along the subsequence.
- Along this sub-subsequence, agent i prefers piece $\pi(i)$ at the corner of τ_t owned by i , and these corners converge to x^* . By closedness, agent i prefers piece $\pi(i)$ at x^* .

Hence, at the cut x^* , the three agents prefer three different pieces. Giving each agent i the piece $\pi(i)$ yields an envy-free division with contiguous pieces. The same argument works for n agents, using Sperner's lemma on Δ_{n-1} . This proves Theorem 3. $\square$

Remarks.

- The proof is *existential*: it relies on a limit, so it does not lead to an efficient algorithm. This is unavoidable, since no finite protocol finds an envy-free contiguous division for $n \geq 3$ (Stromquist 2008).
- However, for a fixed fine triangulation, the walking proof of Sperner's lemma is constructive, and yields an *approximately* envy-free contiguous division.

Exercises

Question 1 (Stromquist's moving-knife procedure, 1980). Consider the following procedure for 3 agents.

- A referee moves a *sword* continuously from 0 to 1; let x be its position.
- At the same time, each agent i holds a *knife* at the point $k_i \in [x, 1]$ with $v_i([x, k_i]) = v_i([k_i, 1])$. Let y be the position of the *middle* knife, and let $L = [0, x]$, $M = [x, y]$, $R = [y, 1]$.
- An agent shouts “stop” as soon as they value L at least as much as both M and R .
- The cake is cut at x and y . The shouter receives L . Of the other two agents, the one whose knife is further left receives M , and the other receives R . (Ties are broken arbitrarily.)

- (a) Prove that the procedure terminates and outputs an envy-free contiguous allocation.

- (b) Now the cake is a *chore*: each agent i has a cost function c_i (normalized, divisible and non-negative, as for goods) and wants to receive a piece of *minimum* cost. An allocation is envy-free if $c_i(A_i) \leq c_i(A_j)$ for all i, j . Design a moving-knife procedure with 2 cuts that gives an envy-free contiguous allocation of the chore among 3 agents, and prove that it is correct.

Question 2 (Sperner's lemma with orientation). Consider a Sperner-coloured triangulation of a triangle whose main vertices v_1, v_2, v_3 appear in counterclockwise order. Call a fully coloured baby triangle *positive* if its colours 1, 2, 3 appear in counterclockwise order, and *negative* otherwise. Prove that

$$\#\{\text{positive baby triangles}\} - \#\{\text{negative baby triangles}\} = 1.$$

In particular, this gives another proof that the number of fully coloured baby triangles is odd.

Hint: walk counterclockwise around the boundary of each baby triangle, and count the edges you traverse from colour 1 to colour 2 minus those you traverse from colour 2 to colour 1. What happens when you sum this over all baby triangles?